PRACTICE PAPERS 2021

www.mtg.in | May 2021 | Pages 80 | ₹ 40

JEE MAIN ADVANCED | BITSAT

CBSE Class XII warm-up!

MATHEMATICS today

India's #1 MATHEMATICS MONTHLY for JEE (Main & Advanced)

OLYMPIAD CORNER

SOLVED PAPER 2021 JEE MAIN

CHALLENGING PROBLEMS

CONCEPT MAP CLASS XI & XII

You Ask ?

We Answer ✓

MATHEMATICS today

Vol. XXXIX No. 5 May 2021

Corporate Office:
Plot 99, Sector 44 Institutional Area,
Gurugram -122 003 (HR), Tel : 0124-6601200
e-mail : info@mtg.in website : www.mtg.in

Regd. Office:
406, Taj Apartment, Near Safdarjung Hospital,
Ring Road, New Delhi - 110029.
Managing Editor : Mahabir Singh
Editor : Anil Ahlawat

CONTENTS

Competition Edge

6 JEE Main
Solved Paper

16 JEE Main
Mock Test Paper

23 Focus JEE Main
Practice Questions

28 JEE Advanced
Practice Paper

37 Olympiad Corner

42 Challenging Problems

48 BITSAT
Practice Paper

75 You Ask We Answer

Class XI

40 Concept Map
Complex Numbers

57 Monthly Test Drive

Class XII

41 Concept Map
Determinants

59 CBSE warm-up!
Practice Paper

71 Monthly Test Drive

Printed and Published by Mahabir Singh on behalf of MTG Learning Media Pvt. Ltd. Printed at HT Media Ltd., B-2, Sector-63, Noida, UP-201307 and published at 406, Taj Apartment, Ring Road, Near Safdarjung Hospital, New Delhi - 110029.
Editor : Anil Ahlawat

SOLVED PAPER

Held on 16th March, Morning Shift

JEE MAIN 2021

SECTION-A (MULTIPLE CHOICE QUESTIONS)

1. If n is the number of irrational terms in the expansion of $\left(3^{\frac{1}{4}}+5^{\frac{1}{8}}\right)^{60}$, then $(n-1)$ is divisible by

(a) 26 (b) 7 (c) 8 (d) 30

2. Let P be a plane $lx + my + nz = 0$ containing the line, $\frac{1-x}{1}=\frac{y+4}{2}=\frac{z+2}{3}$. If plane P divides the line segment AB joining points $A(-3, -6, 1)$ and $B(2, 4, -3)$ in ratio $k : 1$, then the value of k is equal to

(a) 1.5 (b) 4 (c) 3 (d) 2

3. The locus of the midpoints of the chord of the circle, $x^2 + y^2 = 25$ which is tangent to the hyperbola, $\frac{x^2}{9}-\frac{y^2}{16}=1$ is

(a) $(x^2 + y^2)^2 - 9x^2 - 16y^2 = 0$
(b) $(x^2 + y^2)^2 - 9x^2 + 144y^2 = 0$
(c) $(x^2 + y^2)^2 - 16x^2 + 9y^2 = 0$
(d) $(x^2 + y^2)^2 - 9x^2 + 16y^2 = 0$

4. Let $A=\begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$, $i=\sqrt{-1}$. Then, the system of linear equations $A^8\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 8 \\ 64 \end{bmatrix}$ has

(a) A unique solution (b) No solution
(c) Exactly two solutions
(d) Infinitely many solutions

5. The number of elements in the set $\{x \in R : (|x| - 3)|x + 4| = 6\}$ is equal to

(a) 1 (b) 4 (c) 3 (d) 2

6. Let a complex number z, $|z| \neq 1$, satisfy $\log_{\frac{1}{\sqrt{2}}}\left(\frac{|z|+11}{(|z|-1)^2}\right)\le 2$. Then, the largest value of $|z|$ is equal to ______.

(a) 5 (b) 7 (c) 8 (d) 6

7. If for $a > 0$, the feet of perpendiculars from the points $A(a, -2a, 3)$ and $B(0, 4, 5)$ on the plane $lx + my + nz = 0$ are points $C(0, -a, -1)$ and D respectively, then the length of line segment CD is equal to

(a) $\sqrt{31}$ (b) $\sqrt{55}$ (c) $\sqrt{41}$ (d) $\sqrt{66}$

8. Which of the following Boolean expression is a tautology?

(a) $(p \wedge q) \wedge (p \to q)$ (b) $(p \wedge q) \to (p \to q)$
(c) $(p \wedge q) \vee (p \to q)$ (d) $(p \wedge q) \vee (p \vee q)$

9. If for $x \in \left(0, \frac{\pi}{2}\right)$, $\log_{10}\sin x + \log_{10}\cos x = -1$ and $\log_{10}(\sin x + \cos x) = \frac{1}{2}(\log_{10} n - 1)$, $n > 0$, then the value of n is equal to

(a) 9 (b) 20 (c) 16 (d) 12

10. The number of roots of the equation, $(81)^{\sin^2 x}+(81)^{\cos^2 x}=30$ in the interval $[0, \pi]$ is equal to

(a) 4 (b) 3 (c) 8 (d) 2

11. Let a vector $\alpha\hat{i}+\beta\hat{j}$ be obtained by rotating the vector $\sqrt{3}\,\hat{i}+\hat{j}$ by an angle 45° about the origin in counterclockwise direction in the first quadrant. Then the area of triangle having vertices (α, β), $(0, \beta)$ and $(0, 0)$ is equal to

(a) 1 (b) $2\sqrt{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{1}{2}$

12. A pack of cards has one card missing. Two cards are drawn randomly and are found to be spades. The probability that the missing card is not a spade, is

(a) $\frac{39}{50}$ (b) $\frac{52}{867}$ (c) $\frac{3}{4}$ (d) $\frac{22}{425}$

13. Let the position vectors of two points P and Q be $3\hat{i}-\hat{j}+2\hat{k}$ and $\hat{i}+2\hat{j}-4\hat{k}$, respectively. Let R and S be two points such that the direction ratios of lines PR and QS are $(4, -1, 2)$ and $(-2, 1, -2)$, respectively.

Let lines PR and QS intersect at T. If the vector $\overrightarrow{TA}$ is perpendicular to both $\overrightarrow{PR}$ and $\overrightarrow{QS}$ and the length of vector $\overrightarrow{TA}$ is $\sqrt{5}$ units, then the modulus of a position vector of A is

(a) $\sqrt{227}$ (b) $\sqrt{171}$ (c) $\sqrt{5}$ (d) $\sqrt{482}$

14. Let the functions $f: R \to R$ and $g: R \to R$ be defined as

$$f(x)=\begin{cases} x+2, & x<0 \\ x^2, & x \ge 0 \end{cases} \text{ and } g(x)=\begin{cases} x^3, & x<1 \\ 3x-2 & x \ge 1 \end{cases}$$

Then, the number of points in R where $(fog)(x)$ is NOT differentiable is equal to

(a) 3 (b) 0 (c) 2 (d) 1

15. Let $S_k = \sum_{r=1}^{k} \tan^{-1}\left(\frac{6^r}{2^{2r+1}+3^{2r+1}}\right)$.

Then $\lim_{k\to\infty} S_k$ is equal to

(a) $\cot^{-1}\left(\frac{3}{2}\right)$ (b) $\frac{\pi}{2}$

(c) $\tan^{-1}\left(\frac{3}{2}\right)$ (d) $\tan^{-1}(3)$

16. If $y = y(x)$ is the solution of the differential equation, $\frac{dy}{dx} + 2y\tan x = \sin x$, $y\left(\frac{\pi}{3}\right) = 0$, then the maximum value of the function $y(x)$ over R is equal to

(a) $-15/4$ (b) 8 (c) 1/2 (d) 1/8

17. Let $[x]$ denote greatest integer less than or equal to x. If for $n \in N$, $(1-x+x^3)^n = \sum_{j=0}^{3n} a_j x^j$, then

$$\sum_{j=0}^{\left[\frac{3n}{2}\right]} a_{2j} + 4\sum_{j=0}^{\left[\frac{3n-1}{2}\right]} a_{2j+1} \text{ is equal to}$$

(a) 2^{n-1} (b) 1 (c) n (d) 2

18. The range of $a \in R$ for which the function

$f(x) = (4a-3)(x + \log_e 5) + 2(a-7)\cot\left(\frac{x}{2}\right)\sin^2\left(\frac{x}{2}\right)$,

$x \ne 2n\pi$, $n \in N$, has critical point, is

(a) $(-3, 1)$ (b) $\left[-\frac{4}{3}, 2\right)$

(c) $[1, \infty)$ (d) $(-\infty, -1]$

19. Consider three observations a, b, and c such that $b = a + c$. If the standard deviation of $a + 2$, $b + 2$, $c + 2$ is d, then which of the following is true?

(a) $b^2 = 3(a^2 + c^2 + d^2)$ (b) $b^2 = a^2 + c^2 + 3d^2$

(c) $b^2 = 3(a^2 + c^2) - 9d^2$ (d) $b^2 = 3(a^2 + c^2) + 9d^2$

20. If the three normals drawn to the parabola, $y^2 = 2x$ pass through the point $(a, 0)$, $a \ne 0$, then 'a' must be greater than

(a) $-1/2$ (b) 1 (c) -1 (d) 1/2

SECTION-B (NUMERICAL VALUE TYPE)

Attempt any 5 out of 10.

21. If $\lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$, then $a + b + c$ is equal to _____.

22. Let $f: R \to R$ be a continuous function such that $f(x) + f(x+1) = 2$, for all $x \in R$. If $I_1 = \int_0^8 f(x)dx$ and $I_2 = \int_{-1}^{3} f(x)dx$, then the value of $I_1 + 2I_2$ is equal to _____.

23. If the normal to the curve $y(x) = \int_0^x (2t^2 - 15t + 10)dt$ at a point (a, b) is parallel to the line $x + 3y = -5$, $a > 1$, then the value of $|a + 6b|$ is equal to _____.

24. Let $P = \begin{bmatrix} -30 & 20 & 56 \\ 90 & 140 & 112 \\ 120 & 60 & 14 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 7 & \omega^2 \\ -1 & -\omega & 1 \\ 0 & -\omega & -\omega+1 \end{bmatrix}$

where $\omega = \frac{-1+i\sqrt{3}}{2}$, and I_3 be the identity matrix of order 3. If the determinant of the matrix $(P^{-1}AP - I_3)^2$ is $\alpha\omega^2$, then the value of α is equal to _____.

25. Let $ABCD$ be a square of side of unit length. Let a circle C_1 centered at A with unit radius is drawn. Another circle C_2 which touches C_1 and the lines AD and AB are tangent to it, is also drawn. Let a tangent line from the point C to the circle C_2 meet the side AB at E. If the length of EB is $\alpha + \sqrt{3}\beta$, where α, β are integers, then $\alpha + \beta$ is equal to _____.

26. The total number of 3×3 matrices A having entries from the set $\{0, 1, 2, 3\}$ such that the sum of all the diagonal entries of AA^T is 9, is equal to _____.

27. Let z and w be two complex number such that $w = z\bar{z} - 2z + 2$, $\left|\frac{z+i}{z-3i}\right| = 1$ and $\text{Re}(w)$ has minimum value. Then, the minimum value of $n \in N$ for which w^n is real, is equal to _____.

28. Consider an arithmetic series and a geometric series having four initial terms from the set $\{11, 8, 21, 16, 26, 32, 4\}$. If the last terms of these series are the maximum possible four digit numbers, then the number of common terms in these two series is equal to _____.

29. Let $f: (0, 2) \to R$ be defined as

$$f(x) = \log_2\left(1 + \tan\left(\frac{\pi x}{4}\right)\right).$$

Then, $\lim_{n\to\infty} \frac{2}{n}\left(f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + ... + f(1)\right)$ is equal to ____.

30. Let the curve $y = y(x)$ be the solution of differential equation, $\frac{dy}{dx} = 2(x+1)$. If the numerical value of area bounded by the curve $y = y(x)$ and x-axis is $\frac{4\sqrt{8}}{3}$, then the value of $y(1)$ is equal to ____.

SOLUTIONS

1. (a) : We have, $T_{r+1} = {}^{60}C_r\,(3^{1/4})^{60-r}\,(5^{1/8})^r$
$= {}^{60}C_r\,3^{(60-r)/4}\,(5)^{r/8}$

For rational terms, r should be a multiple of 8 and less than 60.

So, r can be 0, 8, 16, ..., 56 *i.e.*, 8 values

$\Rightarrow$ Number of irrational terms $= 61 - 8 = 53$

$\Rightarrow n = 53 \Rightarrow n - 1 = 52$, which is divisible by 26.

2. (d) : The given plane is $lx + my + nz = 0$ ...(i)

and line is

$$\frac{x-1}{-1} = \frac{y-(-4)}{2} = \frac{z-(-2)}{3} \quad \text{...(ii)}$$

Now, plane (i) containing the line (ii), therefore

$-l + 2m + 3n = 0$ ...(iii) and $l - 4m - 2n = 0$ ...(iv)

Solving (iii) and (iv), we get $\frac{l}{-4+12} = \frac{m}{3-2} = \frac{n}{4-2}$

$\Rightarrow l : m : n = 8 : 1 : 2$

So, equation of plane is, $8x + y + 2z = 0$

Now, let the plane divides the line joining A and B at point C in the ratio $k : 1$. Then,

Coordinates of $C \equiv \left(\frac{2k-3}{k+1}, \frac{4k-6}{k+1}, \frac{-3k+1}{k+1}\right)$

which satisfies the equation of plane.

$$\Rightarrow 8\left(\frac{2k-3}{k+1}\right) + \left(\frac{4k-6}{k+1}\right) + 2\left(\frac{-3k+1}{k+1}\right) = 0$$

$\Rightarrow 14k - 28 = 0 \Rightarrow k = 2$

3. (d) : Let (h, k) be the mid-point on the chord of circle $x^2 + y^2 = 25$ with centre $(0, 0)$.

$\therefore$ Equation of chord is

$$hx + ky = h^2 + k^2 \Rightarrow y = -\frac{h}{k}x + \frac{h^2+k^2}{k} \quad \text{...(i)}$$

Now, (i) will be tangent to hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ if $c^2 = a^2m^2 - b^2$

$$\Rightarrow \left(\frac{h^2+k^2}{k}\right)^2 = 9\left(\frac{-h}{k}\right)^2 - (16) \quad \text{[Using (i)]}$$

$\Rightarrow (h^2 + k^2)^2 = 9h^2 - 16k^2$

$\therefore$ Required locus is$(x^2 + y^2)^2 - 9x^2 + 16y^2 = 0$

4. (b) : We have, $A = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$

$$\Rightarrow A^2 = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}\begin{bmatrix} i & -i \\ -i & i \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} = 2^1\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\Rightarrow A^4 = 2^2\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = 4\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = 8\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\Rightarrow A^8 = 64\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 64\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$= 128\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Now, $A^8\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 64 \end{bmatrix} \Rightarrow 128\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 64 \end{bmatrix}$

$\Rightarrow x - y = \frac{1}{16}$ ....(i) $\quad -x + y = \frac{1}{2}$ ....(ii)

Thus, the system of linear equations has no solution

5. (d) : There are three cases arise :

Case I : When $x < -4 \Rightarrow (-x - 3)(-x - 4) = 6$

$\Rightarrow x^2 + 7x + 12 = 6 \Rightarrow x^2 + 7x + 6 = 0$

$\Rightarrow (x + 6)(x + 1) \Rightarrow x = -6$ or -1

But $x < -4$

$\therefore$ $x = -6$ is the solution *i.e.*, one solution

Case II : When $-4 \le x < 0$

$\Rightarrow (-x - 3)(x + 4) = 6 \Rightarrow x^2 + 7x + 18 = 0$

$\because D < 0$

$\therefore$ No solution

Case III : When $x \ge 0$

$\Rightarrow (x - 3)(x + 4) = 6 \Rightarrow x^2 + x - 18 = 0$

$$\Rightarrow x = \frac{-1 \pm \sqrt{73}}{2} \Rightarrow \frac{-1+\sqrt{73}}{2} \quad (\because x \ge 0)$$

So, one solution.

$\therefore$ The elements in the given set is 2.

6. (b) : We have,

$$\log_{\frac{1}{\sqrt{2}}}\left(\frac{|z|+11}{(|z|-1)^2}\right) \le 2 \Rightarrow \frac{|z|+11}{(|z|-1)^2} \ge \left(\frac{1}{\sqrt{2}}\right)^2$$

$\Rightarrow 2(|z| + 11) \ge (|z| - 1)^2 \Rightarrow |z|^2 - 4|z| - 21 \le 0$

$\Rightarrow (|z| - 7)(|z| + 3) \le 0$

$\Rightarrow |z| \le 7 \quad (\because |z|$ can't be negative)

$\therefore$ Maximum value of $|z| = 7$

7. (d) : D.r.'s of line AC are

$< a - 0, -2a + a, 3 + 1 >$ *i.e.*, $< a, -a, 4 >$

$\Rightarrow\ l = a, m = -a, n = 4$

Also, C lies on the given plane

$\therefore\ -am - n = 0 \Rightarrow a^2 = n \Rightarrow a^2 = 4 \Rightarrow a = 2\ (\because a > 0)$

So, equation of plane is

$2x - 2y + 4z = 0 \Rightarrow x - y + 2z = 0$

Let $D(x, y, z)$ be the foot of perpendicular from the point $B(0, 4, 5)$.

Then, D.r.'s of BD are $< 0 - x, 4 - y, 5 - z >$

$= < -x, 4 - y, 5 - z >$

$\Rightarrow\ -x = 1, 4 - y = -1, 5 - z = 2 \Rightarrow x = -1, y = 5, z = 3$

$\therefore$ Coordinates of $D \equiv (-1, 5, 3)$ and that of $C \equiv (0, -2, -1)$

$\therefore$ Length of $CD = \sqrt{1^2 + 7^2 + 4^2} = \sqrt{66}$

8. (b) : $(p \wedge q) \to (p \to q) \equiv \sim(p \wedge q) \vee (\sim p \vee q)$

$\equiv (\sim p \vee \sim q) \vee (\sim p \vee q) \equiv \sim p \vee (\sim q \vee q) \equiv \sim p \vee t \equiv t$

9. (d) : We have, $\log_{10} \sin x + \log_{10} \cos x = -1$

$\Rightarrow\ \log_{10} (\sin x \cdot \cos x) = -1 \quad \sin x \cdot \cos x = \frac{1}{10} \quad ...(i)$

Also, $\log_{10}(\sin x + \cos x) = \frac{1}{2}(\log_{10} n - 1)$

$= \frac{1}{2}(\log_{10} n - \log_{10} 10)$

$\Rightarrow\ 2\log_{10}(\sin x + \cos x) = \log_{10}\left(\frac{n}{10}\right)$

$\Rightarrow\ (\sin x + \cos x)^2 = \frac{n}{10}$

$\Rightarrow\ 1 + 2\sin x \cos x = \frac{n}{10} \Rightarrow 1 + \frac{2}{10} = \frac{n}{10}$ (Using (i))

$\Rightarrow\ n = 12$

10. (a) : We have, $81^{\sin^2 x} + 81^{\cos^2 x} = 30$

$\Rightarrow\ 3^{4\sin^2 x} + \frac{81}{3^{4\sin^2 x}} = 30$

Let $3^{4\sin^2 x} = t \Rightarrow t + \frac{81}{t} = 30 \Rightarrow t^2 - 30t + 81 = 0$

$\Rightarrow\ t = 27$ or $t = 3 \Rightarrow 3^{4\sin^2 x} = 3^3$ or $3^{4\sin^2 x} = 3^1$

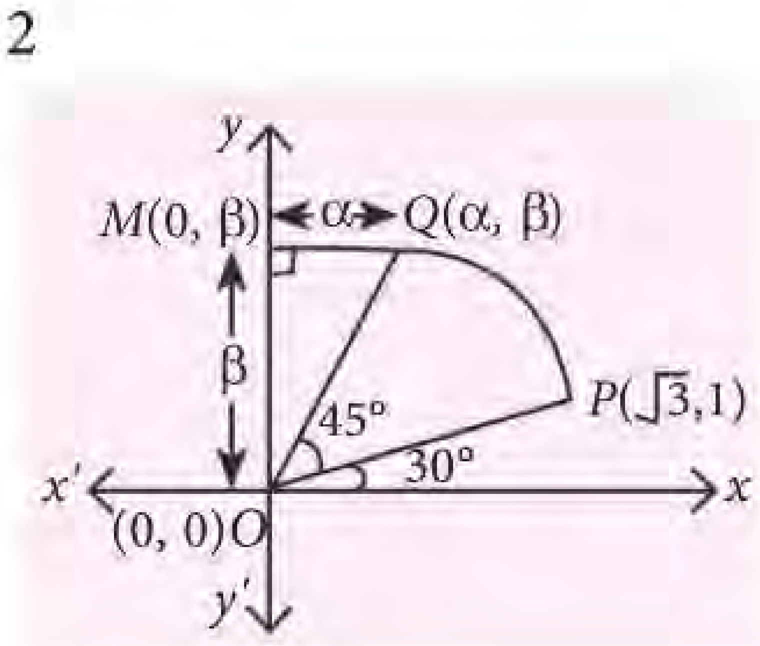
$\Rightarrow\ \sin^2 x = \frac{3}{4}$ or $\sin^2 x = \frac{1}{4} \Rightarrow \sin x = \pm\frac{\sqrt{3}}{2}$ or $\pm\frac{1}{2}$

11. (d) : Let $\overrightarrow{OP} = \sqrt{3}\,\hat{i} + \hat{j}$ and $\overrightarrow{OQ} = \alpha\hat{i} + \beta\hat{j}$

$\therefore\ |\overrightarrow{OP}| = |\overrightarrow{OQ}| = \sqrt{3+1} = 2$

In ΔOMQ, $\frac{\beta}{2} = \cos 15°$

and $\frac{\alpha}{2} = \sin 15° \quad ...(i)$



Area of (ΔOMQ)

$= \frac{1}{2} OM \times MQ$

$= \frac{1}{2}\alpha\beta = \frac{1}{2}(2\cos 15°)(2\sin 15°)$ [Using (i)]

$= 2\sin 15° \cos 15° = \sin 30° = \frac{1}{2}$

12. (a) : Let E_1 be the event that missing card is a spade and E_2 be the event that missing card is not spade. Also, let A be the event of drawing two spades.

$\therefore$ Total probability $= P(E_1)\cdot P(A/E_1) + P(E_2)\cdot P(A/E_2)$

$= \frac{^{13}C_1}{^{52}C_1} \times \frac{^{12}C_2}{^{51}C_2} + \frac{^{39}C_1}{^{52}C_1} \times \frac{^{13}C_2}{^{51}C_2} = \frac{1}{17}$

Now, required probability

$= \frac{P(E_2)\cdot P(A/E_2)}{\text{Total probability}} = \frac{39}{850} \times 17 = \frac{39}{50}$

13. (b) : Equation of line PR is

$\frac{x-3}{4} = \frac{y+1}{-1} = \frac{z-2}{2} = \lambda$ (say)

Equation of line QS is

$\frac{x-1}{-2} = \frac{y-2}{1} = \frac{z+4}{-2} = \mu$ (say)

Any point on line PR is $(4\lambda + 3, -\lambda - 1, 2\lambda + 2)$ and any point on line QS is $(-2\mu + 1, \mu + 2, -2\mu - 4)$.

Since PR and QS intersect at T.

$\therefore\ 4\lambda + 3 = -2\mu + 1, -\lambda - 1 = \mu + 2, 2\lambda + 2 = -2\mu - 4$ for some $\lambda, \mu \in R$

On solving above equations, we get $\lambda = 2, \mu = -5$

$\therefore$ Coordinates of T are $(11, -3, 6)$.

Now, as $\overrightarrow{TA}$ is $\perp$ to both $\overrightarrow{PR}$ and $\overrightarrow{QS}$.

$\therefore\ \overrightarrow{TA} \parallel \overrightarrow{PR} \times \overrightarrow{QS} \Rightarrow \overrightarrow{TA} = m(\overrightarrow{PR} \times \overrightarrow{QS})$

Note that $\overrightarrow{PR} \times \overrightarrow{QS}$ is parallel to $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 2 \\ -2 & 1 & -2 \end{vmatrix}$

$= \hat{i}(2-2) - \hat{j}(-8+4) + \hat{k}(4-2) = 4\hat{j} + 2\hat{k}$

$\therefore\ \overrightarrow{TA} = m(4\hat{j} + 2\hat{k})$

Also, given that $|\overrightarrow{TA}| = \sqrt{5} \Rightarrow |m(4\hat{j} + 2\hat{k})| = \sqrt{5}$

$\Rightarrow\ m^2(16 + 4) = 5 \Rightarrow m^2 = \frac{5}{20} = \frac{1}{4} \Rightarrow m = \pm\frac{1}{2}$

$\therefore\ \overrightarrow{TA} = \pm\frac{1}{2}(4\hat{j} + 2\hat{k})$

$\Rightarrow$ (P.V. of A – P.V. of T) $= \pm\frac{1}{2}(4\hat{j} + 2\hat{k})$

$\Rightarrow$ P.V. of $A = (11\hat{i} - 3\hat{j} + 6\hat{k}) \pm \frac{1}{2}(4\hat{j} + 2\hat{k})$

$= 11\hat{i} - \hat{j} + 7\hat{k}$ or $11\hat{i} - 5\hat{j} + 5\hat{k}$

Required modulus $= \sqrt{121 + 1 + 49} = \sqrt{171}$ units

14. (d) : We have, $f(g(x)) = \begin{cases} g(x) + 2 &, \ g(x) < 0 \\ (g(x))^2 &, \ g(x) \geq 0 \end{cases}$

$$=\begin{cases} x^3+2 & , \quad x<0 \\ x^6 & , \quad 0\le x<1 \\ (3x-2)^2 & , \quad x\ge 1 \end{cases}$$

Now, $(fog)(x)$ is discontinuous at $x = 0$.

$\therefore$ $(fog)(x)$ is non-differentiable at $x = 0$.

For $x = 1$, we have

$$\text{RHD}=\lim_{h\to 0}\frac{f(g(1+h))-f(g(1))}{h}$$

$$=\lim_{h\to 0}\frac{(3(1+h)-2)^2-1}{h}=6$$

$$\text{LHD}=\lim_{h\to 0}\frac{f(g(1-h))-f(g(1))}{-h}=\lim_{h\to 0}\frac{(1-h)^6-1}{-h}=6$$

$\Rightarrow$ RHD = LHD $\Rightarrow f(g(x))$ is differentiable at $x = 1$.

15. (a) : We have, $S_k=\sum_{r=1}^{k}\tan^{-1}\left(\frac{6^r}{2^{2r+1}+3^{2r+1}}\right)$

$$=\sum_{r=1}^{k}\tan^{-1}\left(\frac{3^r\cdot 2^r/2^{2r+1}}{1+(3^{2r+1}/2^{2r+1})}\right)$$

$$=\sum_{r=1}^{k}\tan^{-1}\left(\frac{\frac{3^r}{2^{r+1}}}{1+\left(\frac{3}{2}\right)^r\cdot\left(\frac{3}{2}\right)^{r+1}}\right)$$

$$=\sum_{r=1}^{k}\tan^{-1}\left[\frac{\left(\frac{3}{2}\right)^{r+1}-\left(\frac{3}{2}\right)^r}{1+\left(\frac{3}{2}\right)^r\left(\frac{3}{2}\right)^{r+1}}\right]$$

$$=\tan^{-1}\frac{9}{4}-\tan^{-1}\frac{3}{2}+\tan^{-1}\left(\frac{3}{2}\right)^3-\tan^{-1}\frac{9}{4}$$

$$+\ldots\ldots+\tan^{-1}\left(\frac{3}{2}\right)^{k+1}-\tan^{-1}\left(\frac{3}{2}\right)^k$$

$$\Rightarrow S_k=\tan^{-1}\left(\frac{3}{2}\right)^{k+1}-\tan^{-1}\frac{3}{2}$$

Now, $\lim_{k\to\infty} S_k=\lim_{k\to\infty}\tan^{-1}\left(\frac{3}{2}\right)^{k+1}-\tan^{-1}\frac{3}{2}$

$$=\frac{\pi}{2}-\tan^{-1}\frac{3}{2}=\cot^{-1}\frac{3}{2}$$

16. (d) : The given differential equation is

$$\frac{dy}{dx}+2y\tan x=\sin x,$$

which is a linear differential equation

$$\therefore\ \text{I.F.}=e^{\int 2\tan x\,dx}=e^{-\log\cos^2 x}=\sec^2 x$$

Now, required solution is

$$y\cdot\sec^2 x=\int\frac{\sin x}{\cos^2 x}dx=\int\tan x\cdot\sec x\,dx=\sec x+c$$

$\Rightarrow$ $y = \cos x + c\cos^2 x$ ...(i)

Now, $y\left(\frac{\pi}{3}\right)=0$

$\Rightarrow$ $\cos\frac{\pi}{3}+c\cos^2\frac{\pi}{3}=0\Rightarrow c=-2$ (using (i))

$$\therefore\ y=\cos x-2\cos^2 x=-2\left[\left(\cos x-\frac{1}{4}\right)^2\right]+\frac{1}{8}$$

$$\Rightarrow y_{\max}=\frac{1}{8}$$

17. (b) : We have,

$(1-x+x^3)^n = a_0x^0 + a_1x + a_2x^2 + \ldots\ldots + a_{3n}x^{3n}$ ...(i)

Putting $x = 1$ in (i), we get

$a_0 + a_1 + a_2 + \ldots\ldots + a_{3n} = 1$ ...(ii)

Also, putting $x = -1$ in (i), we get

$a_0 - a_1 + a_2 + a_3 + \ldots\ldots + (-1)^{3n}\cdot a_{3n} = 1$ ....(ii)

Now, adding and subtracting (ii) and (iii), we get

$$2\left\{a_0+a_2+a_4+\ldots..+a_{2\left[\frac{3n}{2}\right]}\right\}=2$$

$$\Rightarrow \sum_{j=0}^{[3n/2]}a_{2j}=1 \quad\ldots\text{(iv)}$$

and $2\left\{a_1+a_3+a_5+\ldots\ldots+a_{2\left[\frac{3n-1}{2}\right]+1}\right\}=0$

$$\Rightarrow \sum_{j=0}^{\left[\frac{3n-1}{2}\right]}a_{2j+1}=0 \quad\ldots\text{(v)}$$

$$\therefore \sum_{j=0}^{[3n/2]}a_{2j}+4\sum_{j=0}^{\left[\frac{3n-1}{2}\right]}a_{2j+1}=1+4\times 0$$

(Using (iv) and (v))

$= 1$

18. (b) : We have,

$$f(x) = (4a-3)(x+\log_e 5)+2(a-7)\cot\left(\frac{x}{2}\right)\cdot\sin^2\left(\frac{x}{2}\right)$$

$$=(4a-3)(x+\log_e 5)+2(a-7)\cdot\cos\frac{x}{2}\cdot\sin\frac{x}{2}$$

$= (4a-3)(x+\log_e 5) + (a-7)\sin x$

Now, $f(x)$ has critical points.

$\therefore$ $f'(x) = 0$

$\Rightarrow$ $(4a-3)+(a-7)\cos x = 0 \Rightarrow \cos x=\frac{3-4a}{a-7}$

Now, $-1 \le \cos x < 1$ $(\because x\ne 2n\pi\Rightarrow \cos x\ne 1)$

$$\Rightarrow -1\le\frac{3-4a}{a-7}<1\Rightarrow\frac{3-4a}{a-7}\ge -1\text{ and }\frac{3-4a}{a-7}<1$$

$$\Rightarrow \frac{3a+4}{a-7}\le 0\text{ and }\frac{5a-10}{a-7}>0$$

$\Rightarrow\ a \in \left[-\frac{4}{3}, 7\right)$ and $a \in (-\infty, 2)$ $\Rightarrow\ a \in \left[-\frac{4}{3}, 2\right)$

19. (c) : As we know that, standard deviation is independent of change of origin.

$\therefore$ S.D. of a, b, c is also d.

$$\Rightarrow\ d^2 = \frac{a^2+b^2+c^2}{3} - \left(\frac{a+b+c}{3}\right)^2$$

$$\Rightarrow\ d^2 = \frac{3(a^2+b^2+c^2)-(2b)^2}{9} \Rightarrow b^2 = 3(a^2+c^2) - 9d^2$$

20. (b) : We have, $y^2 = 2x$

Now, equation of normal to the parabola is

$$y = mx - m - \frac{m^3}{2} = \frac{2mx - 2m - m^3}{2}$$

$\because$ It passes through $(a, 0)$

$\therefore\ 2ma - 2m - m^3 = 0$

$\Rightarrow\ m^3 + 2m(1 - a) = 0$ ...(i)

Let m_1, m_2, m_3 be the roots of the equation (i), then $\Sigma m_1 = 0$, $\Sigma m_1 m_2 = 2(1 - a)$, $m_1 m_2 m_3 = 0$

Now, $m_1^2 + m_2^2 + m_3^2 > 0 \Rightarrow (\Sigma m_1)^2 - 2\,\Sigma m_1 m_2 > 0$

$\Rightarrow\ 0 - 2\,[2(1-a)] > 0 \Rightarrow 4(1-a) < 0 \Rightarrow a > 1$

21. (4) : We have, $\displaystyle\lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x \sin x} = 2$

$$\Rightarrow\ 2 = \lim_{x\to 0} \frac{a\left(1 + x + \frac{x^2}{2!} + \ldots\right) - b\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots\right) + c\left(1 - x + \frac{x^2}{2} - \ldots\ldots\right)}{x^2 \cdot \left(\frac{\sin x}{x}\right)}$$

$$\Rightarrow\ 2 = \lim_{x\to 0} \frac{\left[\left(\frac{a+b+c}{2}\right)x^2 + (a-c)x + (a-b+c) + \ldots\right]}{x^2}$$

For limit to exist, we have

$\frac{a+b+c}{2} = 2 \Rightarrow a+b+c = 4$, $a - c = 0$ and $a - b + c = 0$

22. (16) : We have,

$f(x) + f(x + 1) = 2$ ...(i)

Replacing x with $x + 1$, we get

$f(x + 1) + f(x + 2) = 2$ ...(ii)

From (i) and (ii), we have $f(x) = f(x + 2)$

$\Rightarrow\ f(x)$ is periodic with period 2.

Now, $I_1 = \int_0^8 f(x)dx = 4\int_0^2 f(x)dx$

and $I_2 = \int_{-1}^3 f(x)dx = 2\int_0^2 f(x)\,dx \Rightarrow I_1 = 2I_2$

$\therefore\ I_1 = 4\left[\int_0^1 f(x)dx + \int_1^2 f(x)dx\right]$

$= 4\left[\int_0^1 f(x)dx + \int_0^1 f(x+1)dx\right]$

$= 4\left[\int_0^1 f(x)dx + \int_0^1 (2 - f(x))dx\right]$ [Using (ii)]

$= 4\left[\int_0^1 f(x)dx + \int_0^1 2dx - \int_0^1 f(x)dx\right] = 8$

$\Rightarrow\ I_1 = 8 \Rightarrow I_2 = 4$

$\therefore\ I_1 + 2I_2 = 8 + 8 = 16$

23. (406) : We have, $y(x) = \int_0^x (2t^2 - 15t + 10)dt$

$\Rightarrow\ y'(x) = 2x^2 - 15x + 10$ ....(i)

(Using Leibnitz rule)

Now, slope of normal at $(a, b) = \left.\frac{-1}{y'(x)}\right|_{(a,b)}$

Let $m_1 = \frac{-1}{2a^2 - 15a + 10}$

Now, as normal is parallel to the line $x + 3y = -5$, having slope $(m_2) = \frac{-1}{3}$.

$\therefore\ m_1 = m_2 \Rightarrow 2a^2 - 15a + 10 = 3$

$\Rightarrow\ 2a^2 - 15a + 7 = 0 \Rightarrow (a - 7)(2a - 1) = 0$

$\Rightarrow\ a = 7$ or $a = \frac{1}{2}$ (Neglect) $(\because a > 1)$

From (i), $y(x) = \frac{2x^3}{3} - \frac{15x^2}{2} + 10x$

which passes through (a, b)

$\therefore\ b = \frac{2}{3}(7)^3 - \frac{15}{2}(7)^2 + 10(7) \Rightarrow b = \frac{-413}{6}$

24. (36) : We have, $|(P^{-1}AP - I_3)^2| = |P^{-1}AP - I_3|^2$

$\Rightarrow\ \alpha\omega^2 = |P^{-1}AP - P^{-1}P|^2$ $(\because I = P^{-1}P)$

$= |P^{-1}(AP - P)|^2 = |P^{-1}|^2\ |AP - P|^2$

$= |P^2|^{-1}\,|A - I|^2\,|P|^2 = \frac{1}{|P|^2}|A - I|^2|P|^2$

$\Rightarrow\ \alpha\omega^2 = |A - I|^2$ ...(i)

Now,

$$|A - I| = \begin{vmatrix} 1 & 7 & \omega^2 \\ -1 & -\omega-1 & 1 \\ 0 & -\omega & -\omega \end{vmatrix} = -6\omega\ \ (\because 1 + \omega + \omega^2 = 0)$$

$\Rightarrow\ |A - I|^2 = 36\omega^2 \Rightarrow \alpha\omega^2 = 36\omega^2$ (Using (i))

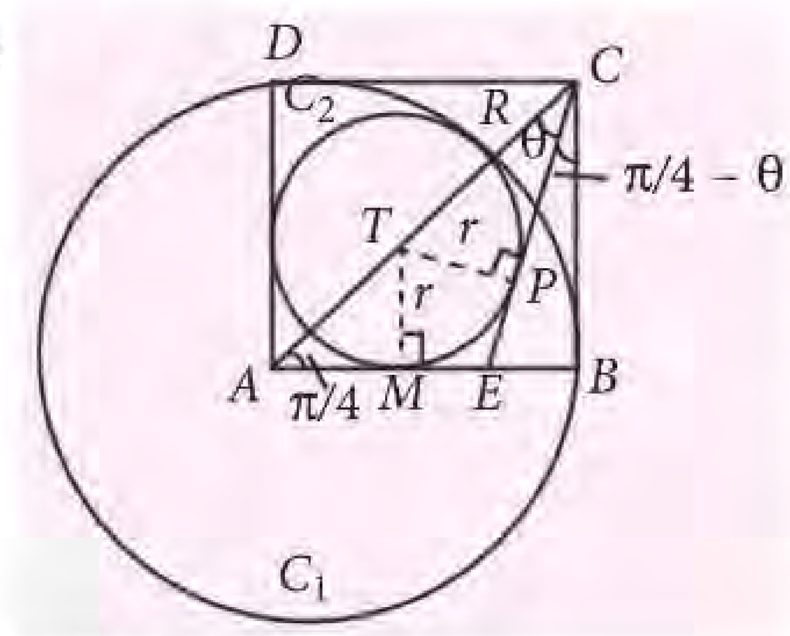
$\Rightarrow\ \alpha = 36$

25. (1) : We have, $AR = 1$ unit (given)

$\Rightarrow\ AT + TR = 1$ ...(i)

Let r be the radius of C_2.

In ΔAMT,



$$AT = \frac{r}{\sin(\pi/4)} = \sqrt{2}r$$

$\therefore$ From (i), we have

$$r\sqrt{2} + r = 1$$

$\Rightarrow$ $r = \sqrt{2} - 1$

Now, $AC = \sqrt{2}$ (diagonal of square)

$\Rightarrow$ $RC = \sqrt{2} - 1 = r \Rightarrow TC = TR + RC = r + r = 2r$

In ΔPCT, we have

$$\sin\theta = \frac{r}{TC} = \frac{r}{2r} = \frac{1}{2} \Rightarrow \theta = 30° \ \therefore \angle ECB = 15°$$

Now, $\tan 15° = \dfrac{EB}{BC} \Rightarrow 2 - \sqrt{3} = \alpha + \sqrt{3}\beta$

$\Rightarrow$ $\alpha = 2, \beta = -1 \ \therefore \alpha + \beta = 1$

26. (766) :

$$\text{Let } A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} a_1 & a_4 & a_7 \\ a_2 & a_5 & a_8 \\ a_3 & a_6 & a_9 \end{bmatrix}$$

Sum of diagonal elements of $AA^T = 9$

$\Rightarrow$ $a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2 + a_8^2 + a_9^2 = 9$

$\because$ All $a_i \in \{0, 1, 2, 3\} \Rightarrow a_i^2 \in \{0, 1, 4, 9\}$

Case-I : $a_i^2 = 9$, then only one $a_i = 3$ and rest will be zero.

$\therefore$ Number of matrices $= {}^9C_1 = 9$

Case II : $a_i^2 = 4$, $a_j^2 = 4$, $a_k^2 = 1$ and rest will be zero.

$\therefore$ Number of matrices $= {}^9C_2 \cdot {}^7C_1 = 252$

Case III : $a_i^2 = 4$ and five a_j's $= 1$ and rest will be zero.

$\therefore$ Number of matrices $= {}^9C_1 \cdot {}^8C_5 = 504$

Case IV : $a_i^2 = 1 \ \forall \ i$

$\therefore$ Number of matrices $= 1$

$\therefore$ Total number of required matrices

$= 9 + 252 + 504 + 1 = 766$

27. (4) : We have, $\left|\dfrac{z+i}{z-3i}\right| = 1 \Rightarrow |z + i| = |z - 3i|$

$\therefore$ Locus of z is the perpendicular bisector of the line segment joining (0, −1) and (0, 3)

$\therefore$ Locus of z is $y = 1$. Let $z = x + i$, $x \in R$

$\Rightarrow$ $w = (x + i)(x - i) - 2(x + i) + 2$

$= (x^2 + 1 - 2x) - 2(i - 1) = [(x - 1)^2 + 2] - 2i$

Now, $\text{Re}(w)$ is minimum $\Rightarrow x - 1 = 0 \Rightarrow x = 1$

$\therefore$ $w = 2(1 - i) = 2\sqrt{2}\, e^{-i\pi/4} \Rightarrow w^n = (2\sqrt{2})^n\, e^{-in\pi/4}$

$\therefore$ Least value of n, for which w^n is real $= 4$

28. (3) : Possible AP is 11, 16, 21, 26 .... with possible last term = 9996

Also, possible G.P. is 4, 8, 16, 32, ... with possible last term = 8192

Now, for common terms, we have

General term of A.P. = General term of G.P.

$\Rightarrow$ $11 + (n - 1)5 = 4(2^{n-1}) \Rightarrow 5n + 6 = 2^{n+1}$

$\Rightarrow$ $n = \dfrac{2^{n+1} - 6}{5}$

This is only possible when, unit digit of 2^{n+1} is 6.

i.e., for n = 3, 7, 11. So, only 3 common terms exist.

29. (1) : Clearly,

$$\lim_{n\to\infty} \frac{2}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \lim_{n\to\infty} \frac{2}{n}\sum_{r=1}^{n} \log_2\left(1 + \tan\left(\frac{\pi r}{4n}\right)\right)$$

$$\Rightarrow \ I = 2\int_0^1 \log_2\left(1 + \tan\frac{\pi}{4}x\right)dx$$

Put $\dfrac{\pi}{4}x = t \Rightarrow dx = \dfrac{4dt}{\pi}$

$$\therefore \ I = \frac{8}{\pi}\int_0^{\pi/4} \log_2(1 + \tan t)dt \qquad \text{...(i)}$$

$$\Rightarrow \ I = \frac{8}{\pi}\int_0^{\pi/4} \log_2\left(1 + \tan\left(\frac{\pi}{4} - t\right)\right)dt$$

$$\Rightarrow \ I = \frac{8}{\pi}\int_0^{\pi/4} \log_2\left[1 + \frac{1 - \tan t}{1 + \tan t}\right]dt$$

$$= \frac{8}{\pi}\int_0^{\pi/4} \log_2\left(\frac{2}{1 + \tan t}\right)dt$$

$$\Rightarrow \ I = \frac{8}{\pi}\int_0^{\pi/4} \log_2 2\, dt - \frac{8}{\pi}\int_0^{\pi/4} \log_2(1 + \tan t)dt \quad \text{....(ii)}$$

Adding (i) and (ii), we get

$$2I = \frac{8}{\pi}\int_0^{\pi/4} \log_2 2dt = \frac{8}{\pi}\,\frac{\pi}{4}$$

($\because$ $\log_a a = 1$, where $a > 0$ and $a \neq 1$)

$\Rightarrow$ $I = 1$

30. (2) : We have, $\dfrac{dy}{dx} = 2(x + 1) \Rightarrow y = (x + 1)^2 + c$

Point of intersection with x-axis $-1 \pm \sqrt{-c} = -1 \pm m$, where $m = \sqrt{-c}$ or $c = -m^2$

Now, area bounded by the curve and x-axis

$$= 2\left|\int_{-1}^{-1+m} ((x+1)^2 + c)\,dx\right| \Rightarrow \frac{2\sqrt{8}}{3} = \left|\left[\frac{(x+1)^3}{3} + cx\right]_{-1}^{-1+m}\right|$$

$$\Rightarrow \ \frac{4\sqrt{2}}{3} = \left|\left[\frac{(x+1)^3 - 3m^2x}{3}\right]_{-1}^{-1+m}\right|$$

$$\Rightarrow \ 4\sqrt{2} = \left|[m^3 - 3m^2(-1 + m)] - [0 + 3m^2]\right|$$

$$\Rightarrow \ 4\sqrt{2} = \left|m^2 + 3m^2 - 3m^3 - 3m^2\right|$$

$\Rightarrow$ $2m^3 = \pm 4\sqrt{2} \Rightarrow m = \pm\sqrt{2} \Rightarrow c = -2$

Thus, $y = (x + 1)^2 - 2 \Rightarrow y(1) = 2$

unacademy

Strategy and tips to crack JEE Advanced in first Attempt with top 100 A.I.R.

Vineet Loomba

IITian | Mentor to 100%iler in both Feb & March Attempts of JEE Main 2021

JEE Advanced is not only an exam but a wish for lakhs of aspirants who dream to enter the **Golden Gates of IITs**, the top most Engineering Institutes of the country. It is believed to be one of the toughest examinations in the world at higher secondary level. But with right strategy and guidance even a **mediocre & hardworking student** can crack the exam with ease. Let's discuss the complete strategy & tips which will help you outshine in this exam.

First step is to choose the **right mentors**. Research, try them yourself through demo classes and then choose your mentors independently. Once chosen stick with them till end. Interact with them regularly, share your problems and find effective solutions.

Second step is to make a good time table. A disciplined preparation is much more fruitful than an undisciplined one. Prepare a time table such that you can follow it around 90%. The remaining 10% portion will give you the necessary push to work harder the next day. Start with 6 hrs daily and move up to 10-14 hrs depending on your capacity. Ensure that you give equal time to Physics, Chemistry and Mathematics. An imbalanced time distribution will cost you dearly in the long run. It is important that all the three subjects are more or less equally strong.

Books and Study Material play an equally important role in your preparation. Consult with your chosen mentors as they will be in the best position to help you choose the best books and study materials for you.

Making **clean, colourful & detailed notes** is another aspect which you should never ignore. Use 3 pens of different colours along with highlighters to make crystal clear notes. Note down all the important points being discussed in the classes by your mentors very carefully.

A key aspect of success for any competitive exam is **Revision**. A well spaced and timely revision can boost your rank by thousands. Don't forget to revise the chapters which you have already covered before jumping onto new chapters. Keep Sundays exclusively for revision. **Formula sheets, flash cards or Mind Maps** are a good source of fast revision many a times during the day.

JEE preparation makes students to sit for long hours during the day. Ensure that you **take breaks** of 10-15 minutes at regular intervals and **stretch your body**. Go for a 20-minute brisk walk during morning and evening hours' daily. Include fruits and juices in your diet along with soaked almonds.

Stay away from the electrons (Negative people) in your friend circle. Be a Proton, work positively and enjoy this journey. Your actions in this journey are going to be life long memories, make sure they are the happy ones.

Vineet Loomba

IITian | Mentor to 100%iler in both Feb & March attempts of JEE Main 2021.

Being a Civil Engineering Graduate from IIT Roorkee, Vineet Loomba Sir gave up a promising career in Civil Engineering to help students prepare for IIT JEE. His experience of more than a decade and his belief in systematic and planned preparation has helped numerous students to achieve their dreams. His friendly approach and unconventional teaching style has made him popular amongst students. Students love him and he loves his students. He is currently No.1 Educator at Unacademy and through this platform, he is impacting the lives of thousands of learners on daily basis.

Scan the QR code to get upto 50% off on Unacademy subscription and start learning from Vineet Loomba

MOCK TEST PAPER 2021

JEE MAIN

Exam Dates 24th to 28th May 2021

Chapters covered : Mathematical Induction, Binomial Theorem, Sequences and Series, Conic Sections, Statistics, Probability, Functions, Limits, Continuity, Differentiability and Co-ordinate Geometry-3D.

SECTION-A (MULTIPLE CHOICE QUESTIONS)

1. $\frac{\frac{1}{2}\cdot\frac{2}{2}}{1^3}+\frac{\frac{2}{2}\cdot\frac{3}{2}}{1^3+2^3}+\frac{\frac{3}{2}\cdot\frac{4}{2}}{1^3+2^3+3^3}+......$upto n terms =

(a) $\frac{n-1}{n}$ (b) $\frac{n}{n+1}$ (c) $\frac{n+1}{n+2}$ (d) $\frac{n+1}{n}$

2. For all $n \in N$, $\cos\theta \cos 2\theta \cos 4\theta \ldots \cos 2^{n-1}\theta$ equals to

(a) $\frac{\sin 2^n\theta}{2^n \sin\theta}$ (b) $\frac{\sin 2^n\theta}{\sin\theta}$

(c) $\frac{\cos 2^n\theta}{2^n \cos 2\theta}$ (d) $\frac{\cos 2^n\theta}{2^n \sin\theta}$

3. If $49^n + 16n + \lambda$ is divisible by 64 for all $n \in N$, then the least negative integral value of λ is

(a) –2 (b) –1 (c) –3 (d) –4

4. Let $P(n) : n^2 + n + 1$ is an even integer. If $P(k)$ is assumed true $\Rightarrow P(k + 1)$ is true. Therefore, $P(n)$ is

(a) true for $n > 1$ (b) true for all $n \in N$

(c) true for $n > 2$ (d) none of these

5. If $-\frac{\pi}{2} < x < \frac{\pi}{2}$, and the sum to infinite terms of the series $\cos x + \frac{2}{3}\cos x \sin^2 x + \frac{4}{9}\cos x \sin^4 x + ...$ is finite, then

(a) $x \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$ (b) $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(c) $x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ (d) none of these

6. If a, x, b are in A.P., a, y, b are in G.P. and a, z, b are in H.P. such that $x = 9z$ and $a > 0, b > 0$, then

(a) $|y| = 3z$ and $x = 3|y|$ (b) $y = 2|z|$ and $|x| = 3y$

(c) $2y = x + z$ (d) none of these

7. If $(a^2, a - 2)$ be a point interior to the region of the parabola $y^2 = 2x$ bounded by the chord joining the points (2, 2) and (8, –4), then the set of all possible values of a, is

(a) $(-2, \sqrt{2})$ (b) $(-3, 2)$

(c) $(-2, 2\sqrt{2})$ (d) $(-2+2\sqrt{2}, 2)$

8. If e_1 is the eccentricity of the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ and e_2 be the eccentricity of the hyperbola passing through the focus of the ellipse such that $e_1e_2 = 1$, then equation of the hyperbola is

(a) $\frac{x^2}{9}-\frac{y^2}{16}=1$ (b) $\frac{x^2}{16}-\frac{y^2}{9}=-1$

(c) $\frac{x^2}{9}-\frac{y^2}{25}=1$ (d) none of these

9. PQ is a double ordinate of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ such that OPQ is an equilateral triangle, O being the centre of the hyperbola, then the eccentricity e of the hyperbola satisfies

(a) $1 < e < \frac{2}{\sqrt{3}}$ (b) $e = \frac{2}{\sqrt{3}}$

(c) $e = \frac{\sqrt{3}}{2}$ (d) $e > \frac{2}{\sqrt{3}}$

10. If $f(x) = \frac{9^x}{9^x+9}$, then

$$f\left(\frac{1}{2019}\right)+f\left(\frac{2}{2019}\right)+f\left(\frac{3}{2019}\right)+\ldots+\left(\frac{4037}{2019}\right)=$$

(a) 1009 (b) $\frac{4037}{2}$ (c) 2018 (d) 2019

11. The probability that $\sin^{-1}(\sin x) + \cos^{-1}(\cos y)$ is an integer $x, y \in \{1, 2, 3, 4\}$, is

(a) $\frac{1}{16}$ (b) $\frac{3}{16}$

(c) $\frac{15}{16}$ (d) none of these.

12. Let the probability P_n that a family has exactly n children be αp^n when $n \geq 1$ and $P_0 = 1 - \alpha p(1 + p + p^2 + ...)$. Suppose that all sex distributions of n children have the same probability. If $k \geq 1$, then the probability that a family contains exactly k boys is

(a) $\frac{2\alpha}{(2-p)^{k+1}}$ (b) $\frac{p^k}{(2-p)^{k+1}}$

(c) $\frac{2\alpha \cdot p}{(2-p)^{k+1}}$ (d) none of these

13. The mean and median of 100 items are 50 and 52 respectively. The value of largest item is 100. It was later found that it is 110 and not 100. The true mean and median are

(a) 50.10, 51.5 (b) 50.10, 52

(c) 50, 51.5 (d) none of these

14. The points $A(5, -1, 1)$, $B(7, -4, 7)$, $C(1, -6, 10)$ and $D(-1, -3, 4)$ are the vertices of a

(a) trapezium (b) rectangle

(c) rhombus (d) square

15. Let $f(x) = \min\{1, \cos x, 1 - \sin x\}$, $-\pi \leq x \leq \pi$. Then, $f(x)$ is

(a) not continuous at $x = \pi/2$

(b) continuous but not differentiable at $x = 0$

(c) neither continuous nor differentiable at $x = \pi/2$

(d) none of these

16. Let f be a differentiable function satisfying the condition:

$$f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)} \text{ for all } x, y \in R (y \neq 0) \text{ and } f(y) \neq 0$$

If $f'(1) = 2$, then $f'(x)$ is equal to

(a) $2f(x)$ (b) $\frac{f(x)}{x}$ (c) $2x f(x)$ (d) $\frac{2f(x)}{x}$

17. Let a function $f(x)$ defined on [3, 6] be given by

$$f(x) = \begin{cases} \log_e[x] & , 3 \leq x < 5 \\ |\log_e x| & , 5 \leq x < 6 \end{cases}$$

Then $f(x)$ is

(a) continuous and differentiable on [3, 6]

(b) continuous on [3, 6) but not differentiable at $x = 4, 5$

(c) differentiable on [3, 6) but not continuous at $x = 4, 5$

(d) none of these

18. If $f(x) = x \dfrac{e^{[x]+|x|} - 2}{[x] + |x|}$, then $\lim_{x \to 0} f(x)$, is

(a) -1 (b) 0

(c) 1 (d) non-existent

19. The value of $\lim_{x \to -\infty} \left\{ \dfrac{x^4 \sin\left(\frac{1}{x}\right) + x^2}{1 + |x^3|} \right\}$ is

(a) 1 (b) -1 (c) 0 (d) ∞

20. If $f(x) = \begin{cases} \dfrac{\sin\{\cos x\}}{x - \frac{\pi}{2}}, & x \neq \frac{\pi}{2} \\ 1, & x = \frac{\pi}{2} \end{cases}$

where $\{\cdot\}$ represents the fractional part function, then $f(x)$ is

(a) continuous at $x = \pi/2$

(b) $\lim_{x \to \pi/2} f(x)$ exists, but $f(x)$ is not continuous at $x = \pi/2$

(c) $\lim_{x \to \pi/2} f(x)$ does not exists

(d) $\lim_{x \to \pi/2^-} f(x) = 1$

SECTION-B (NUMERICAL VALUE TYPE)

Attempt any 5 out of 10.

21. If the coefficients of x and x^2 in the expansion of $(1 + x)^m (1 - x)^n$, $m, n \in N$ are 3 and -6 respectively, then value of m is ______.

22. The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14. Then value of n equals ______.

23. Let $S_1, S_2, ...$ be squares such that for each $n \geq 1$ the length of side of S_n equals the length of diagonal of S_{n+1}. If the length of side of S_1 is 10 cm, then find the least value of n for which the area of S_n less than 1 sq. cm.

24. The area (in sq. units) of the quadrilateral formed by the tangents at the end-points of latusrectum of $\frac{x^2}{9} + \frac{y^2}{5} = 1$ is equal to ______.

25. Let f be a real valued function satisfying $f(x + y) = f(x) f(y)$ for all $x, y \in R$ such that $f(1) = 2$. If

$$\sum_{k=1}^{n} f(a+k) = 16(2^n - 1), \text{ then } a = \_\_\_\_\_\_.$$

26. Let $f(x)$ be a polynomial satisfying

$(f(\alpha))^2 + (f'(\alpha))^2 = 0$. Then, $\lim\limits_{x\to\alpha} \dfrac{f(x)}{f'(x)}\left[\dfrac{f'(x)}{f(x)}\right]$ = ______.

(where $[\cdot]$ denotes the greatest integer function)

27. The first of the two samples has 100 items with mean 15 and standard deviation 3. If the whole group has 250 items with mean 15.6 and standard deviation $\sqrt{13.44}$, then the standard deviation of the second group is ______.

28. The graph of the function $y = f(x)$ has a unique tangent at the point $(a, 0)$ through which the graph passes. Then $\lim\limits_{x\to a} \dfrac{\log_e(1+6f(x))}{3f(x)}$ is ______.

29. In a ΔABC, the mid-point of the sides AB, BC and CA are respectively $(l, 0, 0)$, $(0, m, 0)$ and $(0, 0, n)$. Then $\dfrac{AB^2 + BC^2 + CA^2}{l^2 + m^2 + n^2}$ = ______.

30. An aeroplane flies around a square, the sides of which measure 100 miles each. The aeroplane covers at a speed of 100 mph the first side, at 200 mph the second side, at 300 mph the third side and 400 mph the fourth side. The average speed of the aeroplane around the square is ______ mph.

SOLUTIONS

1. **(b) :** We have, $T_n = \dfrac{n(n+1)}{4\Sigma n^3} = \dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$

$\therefore\ T_1 = \dfrac{1}{1} - \dfrac{1}{2}, T_2 = \dfrac{1}{2} - \dfrac{1}{3}, \ldots\ldots, T_n = \dfrac{1}{n} - \dfrac{1}{n+1}$

Adding above equations, we get

$$S_n = T_1 + T_2 + \ldots.. + T_n = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

2. **(a) :** Let $P(n) = \cos\theta \cos 2\theta \ldots.. \cos 2^{n-1}\theta = \dfrac{\sin 2^n\theta}{2^n \sin\theta}$

$P(1) = \cos\theta = \dfrac{\sin 2\theta}{2\sin\theta} = \cos\theta$, which is true

$$P(n+1) = (\cos\theta \cos 2\theta \cos 2^2\theta \ldots.. \cos 2^{n-1}\theta) \times \cos 2^n\theta$$

$$= \frac{\sin 2^n\theta}{2^n \sin\theta} \times \cos 2^n\theta = \frac{\sin 2\cdot 2^n\theta}{2\cdot 2^n \sin\theta} = \frac{\sin 2^{n+1}\theta}{2^{n+1}\sin\theta}$$

$\therefore$ By P.M.I., $P(n)$ is true for all $n \in N$.

3. **(b) :** For $n = 1$, we have

$49^n + 16n + \lambda = 49 + 16 + \lambda = 65 + \lambda = 64 + (\lambda + 1)$

This is divisible by 64 if $\lambda = -1$.

For $n = 2$, we have

$49^n + 16n + \lambda = 49^2 + 16 \times 2 + \lambda = 2433 + \lambda$
$= 64 \times 38 + (\lambda + 1)$

This is divisible by 64 if $\lambda = -1$. Hence, $\lambda = -1$

4. **(d) :** Given, $P(n) : n^2 + n + 1$ is an even integer.

For $n = 1$, $P(1) = 3$, which is not an even integer.

$\therefore$ $P(1)$ is not true.

Also, $n^2 + n + 1 = n(n + 1) + 1$ is always an odd integer.

(principle of induction is not applicable)

5. **(b) :** The given infinite series is

$$\cos x + \frac{2}{3}\cos x \sin^2 x + \frac{4}{9}\cos x \sin^4 x + \ldots$$

The above series in an infinite G.P. with common ratio $\dfrac{2}{3}\sin^2 x$.

The sum of the series will exist when $\left|\dfrac{2}{3}\sin^2 x\right| < 1$.

Clearly, $\left|\dfrac{2}{3}\sin^2 x\right| = \dfrac{2}{3}|\sin x|^2 < \dfrac{2}{3} < 1$.

Therefore, the sum of the series exists and given by

$$S = \frac{\cos x}{1 - \frac{2}{3}\sin^2 x} = \frac{3\cos x}{2 + \cos 2x}$$

As, $\cos x$ and $2 + \cos 2x$ are finite for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.

So, sum of the given series is finite for $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

6. **(a) :** Since, a, x, b are in A.P. ; a, y, b are in G.P. and a, z, b are in H.P. , we have x, y, z as A.M., G.M. and H.M. of a and b respectively. Also A.M., G.M. and H.M. are in G.P.

$\therefore\ y^2 = xz \Rightarrow y^2 = 9z^2$ $\quad [\because x = 9z]$

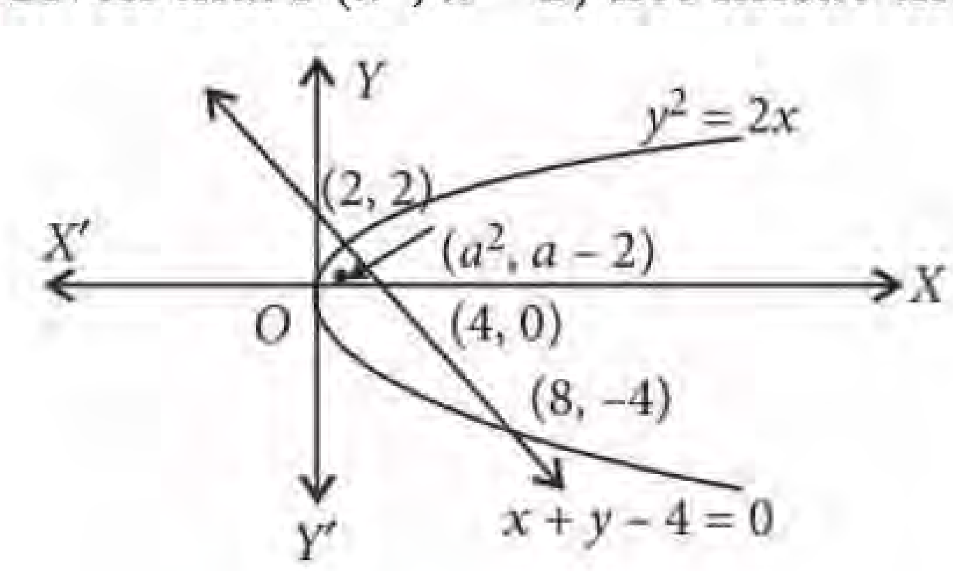
$\Rightarrow y = 3z \Rightarrow |y| = 3z$ $\quad [\because y > 0 \therefore |y| = y]$

Again, $y^2 = xz$ and $x = 9z$

$$\Rightarrow y^2 = \frac{x^2}{9} \Rightarrow 9y^2 = x^2$$

$\Rightarrow 3y = x \Rightarrow 3|y| = x$ $\quad [\because y > 0 \ \therefore |y| = y]$

7. **(d) :** Given that $P(a^2, a - 2)$ lies inside the parabola,



$\therefore\ (a-2)^2 - 2\times a^2 < 0 \Rightarrow a^2 - 4a + 4 - 2a^2 < 0$

$\Rightarrow -a^2 - 4a + 4 < 0 \Rightarrow a^2 + 4a - 4 > 0$

$\Rightarrow (a+2)^2 - (2\sqrt{2})^2 > 0$

$\Rightarrow a+2 < -2\sqrt{2}$ or $a+2 > 2\sqrt{2}$ ...(i)

Since point $P(a^2, a-2)$ and origin $O(0, 0)$ are on the same side of the chord joining (2, 2) and (8, –4).

$\therefore\ (0+0-4)(a^2+a-2-4) > 0$

$\Rightarrow a^2 + a - 6 < 0 \Rightarrow (a+3)(a-2) < 0$

$\Rightarrow -3 < a < 2$ ...(ii)

Also, $-4 < a-2 < 2$ and $0 < a^2 < 8$

$\Rightarrow -2 < a < 2\sqrt{2}$ ...(iii)

From (i), (ii), and (iii), we get that $a \in (-2+2\sqrt{2}, 2)$.

8. (b) : We have, $e_1 = \sqrt{1-\frac{16}{25}} = \frac{3}{5}$

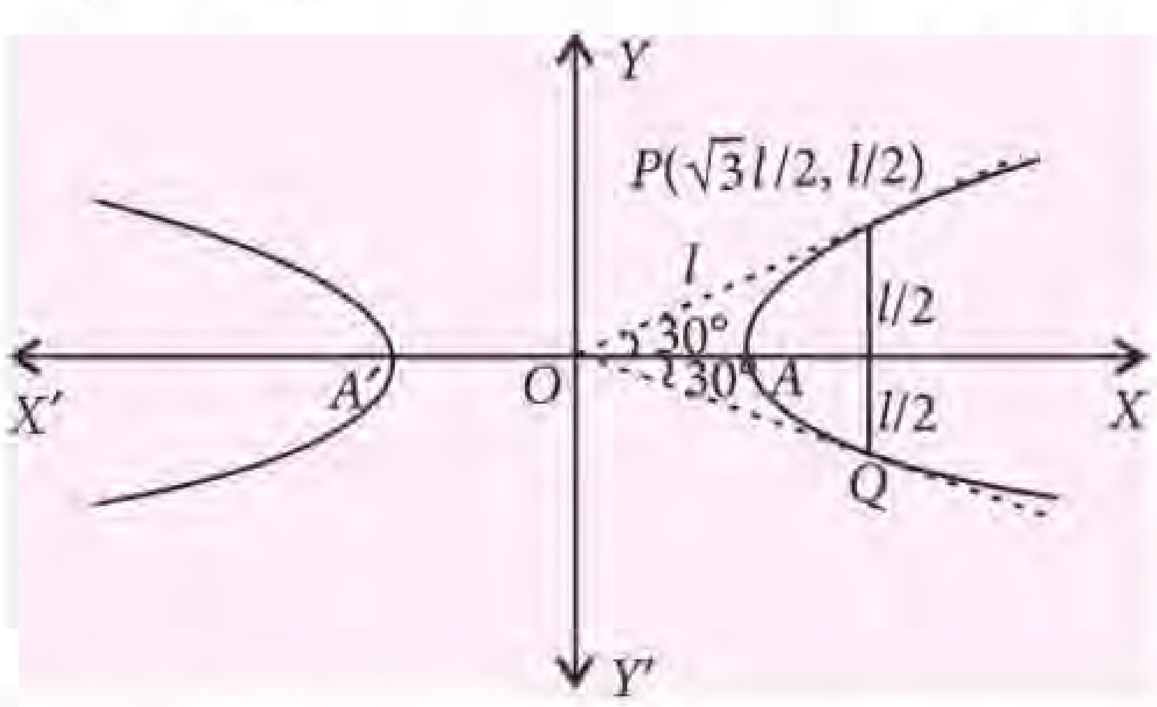
$\because\ e_1e_2 = 1 \Rightarrow e_2 = \frac{5}{3}$

The coordinates of foci of the ellipse are (0, ±3).

Clearly, hyperbola in option (b) passes through (0, ±3) and has eccentricity 5/3.

9. (d) : Let the length of each side of the equilateral triangle *OPQ* be *l* units. Then, the coordinates of *P* are $\left(\frac{\sqrt{3}l}{2}, \frac{l}{2}\right)$.

Point $P\left(\frac{\sqrt{3}l}{2}, \frac{l}{2}\right)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.



$\therefore\ \frac{3l^2}{4a^2} - \frac{l^2}{4b^2} = 1 \Rightarrow (3b^2 - a^2)l^2 = 4a^2b^2$

$\Rightarrow (3e^2-4)l^2 = 4a^2(e^2-1) \Rightarrow l = 2a\sqrt{\frac{e^2-1}{3e^2-4}}$

Since, *l* is real and *e* > 1

$\therefore\ 3e^2 - 4 > 0 \Rightarrow e^2 > \frac{4}{3} \Rightarrow e > \frac{2}{\sqrt{3}}$.

10. (b) : We have $f(x) = \frac{9^x}{9^x+9}$

$\therefore\ f(2-x) = \frac{9^{2-x}}{9^{2-x}+9}$

So, $f(x) + f(2-x) = \frac{9^x}{9^x+9} + \frac{9^{2-x}}{9^{2-x}+9} = \frac{9^x}{9^x+9} + \frac{9}{9^x+9} = 1$

$\Rightarrow f(x) + f(2-x) = 1$

$\therefore\ f\left(\frac{1}{2019}\right) + f\left(\frac{2}{2019}\right) + f\left(\frac{3}{2019}\right) + \ldots + f\left(\frac{4037}{2019}\right)$

$$= \left\{f\left(\frac{1}{2019}\right) + f\left(\frac{4037}{2019}\right)\right\} + \left\{f\left(\frac{2}{2019}\right) + f\left(\frac{4036}{2019}\right)\right\} + \ldots + \left\{f\left(\frac{2018}{2019}\right) + f\left(\frac{2020}{2019}\right)\right\} + f\left(\frac{2019}{2019}\right)$$

$= \{1+1+\ldots+1(2018\text{ times})\} + f(1) = 2018 + \frac{1}{2} = \frac{4037}{2}$

11. (b) : Clearly x should lie in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and y in $[0, \pi]$ in order to get the integer value of $\sin^{-1}(\sin x) + \cos^{-1}(\cos y)$.

$\Rightarrow x = 1$ and $y = 1, 2, 3$

$\therefore$ Required probability $= \frac{3}{16}$.

12. (d) : We are given that $P_n = \alpha p^n,\ n \ge 1$

and $P_0 = 1 - \alpha p(1 + p + p^2 + \ldots\ldots)$.

Now let us define the events in the following way :

E_j = There are j children in the family, $j = 0, 1, 2, \ldots, n$

A = There are exactly k boys in the family

We have, $P(E_j) = P_j = \alpha p^j;\ j = 0, 1, 2, \ldots\ldots, n$

and $P(A\,|\,E_j) = \frac{{}^jC_k}{2^j},\ j \ge k$

Now, $A = \bigcup_{j=k}^{\infty}(A \cap E_j) \Rightarrow P(A) = P\left(\bigcup_{j=k}^{\infty}(A\cap E_j)\right)$

$\therefore\ P(A) = \sum_{j=k}^{\infty} P(A\cap E_j) = \sum_{j=k}^{\infty} P(E_j)P(A\,|\,E_j)$

$= \sum_{j=k}^{\infty} \alpha p^j\left(\frac{{}^jC_k}{2^j}\right) = \alpha\sum_{j=k}^{\infty}\left(\frac{p}{2}\right)^j \cdot {}^jC_K$

$= \alpha\sum_{r=0}^{\infty} {}^{k+r}C_r\left(\frac{p}{2}\right)^{k+r} = \alpha\left(\frac{p}{2}\right)^k \sum_{r=0}^{\infty} {}^{k+r}C_r\left(\frac{p}{2}\right)^r$

13. (b) : Here $n = 100$, mean = 50, median = 52

$\therefore\ \bar{x} = \frac{1}{n}\sum_{i=1}^{100} x_i = 50 \Rightarrow \sum_{i=1}^{100} x_i = 5000$

Now, correct $\sum_{i=1}^{100} x_i = 5000 - 100 + 110 = 5010$

$\therefore$ Correct mean $= \frac{1}{100}\sum_{i=1}^{100} x_i = \frac{5010}{100} = 50.10$

As median is positional average therefore it will remain same.

14. (c) : It can easily be seen that $AB = BC = CD = DA = 7$. So, $ABCD$ is a rhombus or square.

Also, $AC = \sqrt{122}$; $BD = \sqrt{74}$

$\because$ Diagonals are not equal

$\therefore$ $ABCD$ is a rhombus.

15. (b) : Given that $f(x) = \min\{1, \cos x, 1 - \sin x\}$, $-\pi \le x \le \pi$

$$\Rightarrow f(x) = \begin{cases} \cos x & , -\frac{\pi}{2} \le x \le 0 \\ 1 - \sin x & , 0 < x \le \frac{\pi}{2} \\ \cos x & , \frac{\pi}{2} < x \le \pi \end{cases}$$

Now $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \cos x = 1$

and $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (1 - \sin x) = 1$

and, $f(0) = \cos 0 = 1$

Clearly $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x) = f(0)$, so $f(x)$ is continuous at $x = 0$.

Again $Lf'(0) = \frac{d}{dx}(\cos x)\Big|_{x=0} = 0$

and $Rf'(0) = \frac{d}{dx}(1 - \sin x)\Big|_{x=0} = -1$

$\therefore$ $Lf'(0) \ne Rf'(0)$

Hence, $f(x)$ is not differentiable at $x = 0$.

Thus $f(x)$ is continuous at $x = 0$ but not differentiable at $x = 0$.

16. (d) : We have,

$f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$ for all $x, y \in R$ $(y \ne 0)$ and $f(y) \ne 0$

$f(1) = \frac{f(1)}{f(1)} \Rightarrow f(1) = 1.$ [Replacing x and y both by 1]

Now, $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$\Rightarrow f'(x) = f(x) \lim_{h \to 0} \left\{ \frac{\frac{f(x+h)}{f(x)} - 1}{h} \right\}$$

$$\Rightarrow f'(x) = f(x) \lim_{h \to 0} \left\{ \frac{f\left(\frac{x+h}{x}\right) - 1}{h} \right\} \left[\because f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}\right]$$

$$\Rightarrow f'(x) = f(x) \lim_{h \to 0} \left\{ \frac{f\left(1 + \frac{h}{x}\right) - f(1)}{h} \right\} [\because f(1) = 1]$$

$$\Rightarrow f'(x) = \frac{f(x)}{x} \lim_{h \to 0} \left\{ \frac{f\left(1 + \frac{h}{x}\right) - f(1)}{\frac{h}{x}} \right\}$$

$$\Rightarrow f'(x) = \frac{f(x)}{x} f'(1) = \frac{2f(x)}{x} \qquad [\because f'(1) = 2]$$

17. (d) : We have, $f(x) = \begin{cases} \log_e 3 & , 3 \le x < 4 \\ \log_e 4 & , 4 \le x < 5 \\ \log_e x & , 5 \le x < 6 \end{cases}$

Clearly, $f(x)$ continuous and differentiable on $[3, 4) \cup (4, 5) \cup (5, 6)$

At $x = 4$, we have

$\lim\limits_{x \to 4^-} f(x) = \log_e 3$ and $\lim\limits_{x \to 4^+} f(x) = \log_e 4$

$\because$ $\lim\limits_{x \to 4^-} f(x) \ne \lim\limits_{x \to 4^+} f(x)$

Thus, $f(x)$ is neither continuous nor differentiable at $x = 4$.

At $x = 5$, we have

$\lim\limits_{x \to 5^-} f(x) = \log_e 4$ and $\lim\limits_{x \to 5^+} f(x) = \log_e 5$

$\therefore$ $\lim\limits_{x \to 5^-} f(x) \ne \lim\limits_{x \to 5^+} f(x)$

So, $f(x)$ is neither continuous nor differentiable at $x = 5$.

18. (d) : We have, $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} x \frac{e^{-1-x} - 2}{-1-x}$

[$\because$ $[x] = -1$ and $|x| = -x$ when $-1 < x < 0$]

$$\Rightarrow \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 0 \times \frac{e^{-1} - 2}{-1} = 0$$

and, $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} x\left(\frac{e^x - 2}{x}\right)$

[$\because$ $[x] = 0$ and $|x| = x$ when $0 \le x < 1$]

$$\Rightarrow \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^x - 2 = 1 - 2 = -1$$

$\therefore$ $\lim\limits_{x \to 0} f(x)$ does not exist.

19. (b) : We have, $\lim\limits_{x \to -\infty} \left\{ \frac{x^4 \sin\left(\frac{1}{x}\right) + x^2}{1 + |x^3|} \right\}$

$$= \lim_{h \to \infty} \frac{h^4 \sin\left(-\frac{1}{h}\right) + h^2}{1 + |-h^3|}, \quad \text{where } h = -x$$

$$= \lim_{h\to\infty} \frac{-h^4 \sin\left(\frac{1}{h}\right)+h^2}{1+h^3}$$

$$= \lim_{h\to\infty} \frac{-h\sin\left(\frac{1}{h}\right)+\frac{1}{h}}{\frac{1}{h^3}+1} = \frac{-1+0}{0+1} = -1$$

20. (b) : We have,

$$\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{h\to 0} f\left(\frac{\pi}{2}-h\right) = \lim_{h\to 0} \frac{\sin\left\{\cos\left(\frac{\pi}{2}-h\right)\right\}}{-h}$$

$$= \lim_{h\to 0} \frac{\sin\{\sin h\}}{-h} = -\lim_{h\to 0} \frac{\sin(\sin h)}{\sin h} \times \frac{\sin h}{h} = -1$$

and $$\lim_{x\to\frac{\pi}{2}^+} f(x) = \lim_{h\to 0} f\left(\frac{\pi}{2}+h\right) = \lim_{h\to 0} \frac{\sin\left\{\cos\left(\frac{\pi}{2}+h\right)\right\}}{\frac{\pi}{2}+h-\frac{\pi}{2}}$$

$$= \lim_{h\to 0} \frac{\sin\{-\sin h\}}{h} = -\lim_{h\to 0} \frac{\sin(\sin h)}{\sin h} \times \frac{\sin h}{h} = -1$$

So, $\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^+} f(x) \neq f\left(\frac{\pi}{2}\right)$

Therefore, $\lim_{x\to\pi/2} f(x)$ exists but $f(x)$ is not continuous at $x = \pi/2$.

21. (12) : We have, $(1 + x)^m (1 - x)^n$

$$= ({}^mC_0 + {}^mC_1x + {}^mC_2x^2 + \ldots + {}^mC_mx^m) \times ({}^nC_0 - {}^nC_1x + {}^nC_2x^2 + \ldots + (-1)^n\, {}^nC_nx^n)$$

$$= ({}^mC_0\, {}^nC_0) - ({}^mC_0\, {}^nC_1 - {}^nC_0\, {}^mC_1)x + ({}^mC_0\, {}^nC_2 + {}^nC_0\, {}^mC_2 - {}^mC_1\, {}^nC_1)x^2 + \ldots$$

Now, according to problem, we have

$$-({}^mC_0\, {}^nC_1 - {}^nC_0\, {}^mC_1) = 3$$

Also, $({}^mC_0\, {}^nC_2 + {}^nC_0\, {}^mC_2 - {}^mC_1\, {}^nC_1) = -6$

$\Rightarrow$ $m - n = 3$ and $n(n - 1) + m(m - 1) - 2mn = -12$

$\Rightarrow$ $m - n = 3$ and $(m - n)^2 - (m + n) = -12$

$\Rightarrow$ $m - n = 3$ ...(i)

and $m + n = 21$ ...(ii)

Solving (i) and (ii), we get $m = 12$, $n = 9$.

22. (6) : Let the three consecutive terms in the expansion be r^{th}, $(r + 1)^{\text{th}}$, $(r + 2)^{\text{th}}$.

Given that, ${}^{n+5}C_{r-1} : {}^{n+5}C_r : {}^{n+5}C_{r+1} = 5:10:14$

$\Rightarrow$ $\frac{{}^{n+5}C_r}{{}^{n+5}C_{r-1}} = \frac{10}{5}$ and $\frac{{}^{n+5}C_{r+1}}{{}^{n+5}C_r} = \frac{14}{10}$

$\Rightarrow$ $\frac{n+5-r+1}{r} = 2$ and $\frac{n+5-r}{r+1} = \frac{7}{5}$

$\Rightarrow$ $n - 3r + 6 = 0$ ...(i)

and $5n - 12r + 18 = 0$ ...(ii)

Solving (i) and (ii), we get, $n = 6$, $r = 4$.

23. (8) : Length of a side of S_n = Length of a diagonal of S_{n+1}

$\Rightarrow$ Length of a side of $S_n = \sqrt{2}$ (Length of a side of S_{n+1})

$\Rightarrow$ $\frac{\text{Length of a side of } S_{n+1}}{\text{Length of side of } S_n} = \frac{1}{\sqrt{2}}$ for all $n \geq 1$

$\Rightarrow$ Sides of $S_1, S_2, \ldots, S_n$ forms a G.P. with common ratio $\frac{1}{\sqrt{2}}$ and first term 10.

$\therefore$ Length of the side of $S_n = 10\left(\frac{1}{\sqrt{2}}\right)^{n-1} = \frac{10}{2^{(n-1)/2}}$

Now, area of $S_n = (\text{side})^2 = \left(\frac{10}{2^{(n-1)/2}}\right)^2 = \frac{100}{2^{n-1}}$

For area of $S_n < 1$ $\Rightarrow$ $\frac{100}{2^{n-1}} < 1$ $\Rightarrow$ $2^{n-1} > 100$

$\Rightarrow$ $n - 1 \geq 7$ $\Rightarrow$ $n \geq 8$.

24. (27) : The equation of the ellipse is $\frac{x^2}{9} + \frac{y^2}{5} = 1$

$\therefore$ Eccentricity $(e) = \sqrt{1-\frac{5}{9}} = \frac{2}{3}$.

The coordinates of the end-points of latusrectum are

$$L\left(2, \frac{5}{3}\right), L'\left(2, -\frac{5}{3}\right), M\left(-2, \frac{5}{3}\right), M'\left(-2, -\frac{5}{3}\right).$$

The equations of tangents at these points are

$2x + 3y - 9 = 0 \Rightarrow y = \frac{-2}{3}x + 3$ ...(i)

$2x - 3y - 9 = 0 \Rightarrow y = \frac{2}{3}x - 3$ ...(ii)

$-2x + 3y - 9 = 0 \Rightarrow y = \frac{2}{3}x + 3$ ...(iii)

$2x + 3y + 9 = 0 \Rightarrow y = \frac{-2}{3}x - 3$ ...(iv)

Clearly, the above tangents form a parallelogram whose area is given by

$\because$ Area of $||^{\text{gm}} = \left|\frac{(c_1 - c_2)(d_1 - d_2)}{(m_1 - m_2)}\right|$

$\therefore$ $A = \left|\frac{(3+3)\times(-3-3)}{-\frac{2}{3}-\frac{2}{3}}\right| = \frac{6\times 6}{4/3} = 27$ sq.units.

25. (3) : We have, $f(x)=[f(1)]^x=2^x$ for all $x\in R$.

[If $f: R\to R$ is a function satisfying $f(x+y)=f(x)\,f(y)$ for all $x, y\in R$, then $f(x)=[f(1)]^x\ \forall\ x\in R$]

$$\therefore\ \sum_{k=1}^{n} f(a+k)=16(2^n-1)$$

$$\Rightarrow\ \sum_{k=1}^{n} 2^{a+k}=16(2^n-1)\ \Rightarrow\ 2^a\sum_{k=1}^{n}2^k=16(2^n-1)$$

$$\Rightarrow\ 2^a(2+2^2+2^3+\ldots+2^n)=16(2^n-1)$$

$$\Rightarrow\ 2^a\times 2\left(\frac{2^n-1}{2-1}\right)=16(2^n-1)\Rightarrow 2^{a+1}=2^4\ \Rightarrow\ a=3$$

26. (1) : We are given that the polynomial $f(x)$ satisfies the relation $(f(\alpha))^2+(f'(\alpha))^2=0$

$\therefore\ f(\alpha)=0=f'(\alpha)$

$\Rightarrow\ x=\alpha$ is a root of $f(x)$ and $f'(x)$

$\Rightarrow\ (x-\alpha)^2$ is a factor of $f(x)$

Let $f(x)=(x-\alpha)^2\,\phi(x)$.

Then $f'(x)=2(x-\alpha)\phi(x)+(x-\alpha)^2\phi'(x)$.

$$\therefore\ \frac{f(x)}{f'(x)}=\frac{(x-\alpha)\phi(x)}{2\phi(x)+(x-\alpha)\phi'(x)}$$

Now, $\lim\limits_{x\to\alpha}\dfrac{f(x)}{f'(x)}\left[\dfrac{f'(x)}{f(x)}\right]$

$$=\lim_{x\to\alpha}\frac{f(x)}{f'(x)}\left(\frac{f'(x)}{f(x)}-\left\{\frac{f'(x)}{f(x)}\right\}\right),\ [\text{since } [x]=x-\{x\}]$$

$$=\lim_{x\to\alpha}\frac{f(x)}{f'(x)}\times\frac{f'(x)}{f(x)}-\lim_{x\to\alpha}\frac{f(x)}{f'(x)}\left\{\frac{f'(x)}{f(x)}\right\}=1-0=1$$

27. (4) : We have $n_1=100, \bar{x}_1=15, \sigma_1=3$,

$n_1+n_2=250, \bar{x}=15.6$ and $\sigma=\sqrt{13.44}$

We have to find σ_2.

Now, $n_2=250-100=150$

We know that, $\bar{x}=\dfrac{n_1\bar{x}_1+n_2\bar{x}_2}{n_1+n_2}$

$$\Rightarrow\ 15.6=\frac{100\times 15+150\times\bar{x}_2}{250}\ \Rightarrow\ \bar{x}_2=16$$

Hence $d_1=\bar{x}_1-\bar{x}=15-15.6=-0.6$

and $d_2=\bar{x}_2-\bar{x}=16-15.6=0.4$

The variance σ^2 of the combined group is given by the formula

$$(n_1+n_2)\sigma^2=n_1(\sigma_1^2+d_1^2)+n_2(\sigma_2^2+d_2^2)$$

$$\Rightarrow\ 250\times 13.44=100(9+0.36)+150(\sigma_2^2+0.16)$$

$$\Rightarrow\ 150\sigma_2^2=250\times 13.44-100\times 9.36-150\times 0.16=2400$$

$$\Rightarrow\ \sigma_2^2=\frac{2400}{150}=16\Rightarrow\sigma_2=\sqrt{16}=4$$

28. (2) : According to the problem we have, $f(a)=0$ and $f(x)$ is differentiable at $x=a$.

$$\therefore\ \lim_{x\to a}\frac{\log_e(1+6f(x))}{3f(x)}=\lim_{x\to a}\frac{\dfrac{6f'(x)}{1+6f(x)}}{3f'(x)}\ \text{[By LH rule]}$$

$$=\frac{2}{1+6f(a)}\ \text{[Since } f(x) \text{ is passes through } (a, 0)]$$

$= 2$ [As $f(a)=0$]

29. (8) : The coordinates of A, B and C are $A(l, -m, n)$, $B(l, m, -n)$ and $C(-l, m, n)$

$$\therefore\ \frac{AB^2+BC^2+CA^2}{l^2+m^2+n^2}=\frac{4(m^2+n^2)+4(l^2+n^2)+4(l^2+m^2)}{l^2+m^2+n^2}$$

$$=\frac{8(l^2+m^2+n^2)}{l^2+m^2+n^2}=8$$

30. (192) : Total distance covered $= 4\times 100 = 400$ miles

$$\text{Total time taken}=\frac{100}{100}+\frac{100}{200}+\frac{100}{300}+\frac{100}{400}=\frac{25}{12}\ \text{hours}$$

$$\therefore\ \text{Average speed}=\frac{\text{Total distance covered}}{\text{Total time taken}}$$

$$=\frac{400}{25}\times 12=192\text{ mph}$$

FOCUS

Exam Dates 2021

24th to 28th May

JEE MAIN 2021

PRACTICE QUESTIONS

MULTIPLE CHOICE QUESTIONS

1. Which of the following is negative?

(a) $\cos(\tan^{-1}(\tan 4))$ (b) $\sin(\cot^{-1}(\cot 4))$

(c) $\tan(\cos^{-1}(\cos 5))$ (d) $\cot(\sin^{-1}(\sin 4))$

2. If in a right angled triangle ABC, $4 \sin A \cos B - 1 = 0$ and $\tan A$ is real, then A, B, C are in

(a) A.P. (b) G.P.

(c) H.P. (d) none of these

3. If α and β are roots of the equation $ax^2 + bx + c = 0$, then roots of the equation

$a(2x + 1)^2 - b(2x + 1)(3 - x) + c(3 - x)^2 = 0$ are

(a) $\frac{2\alpha+1}{\alpha-3}, \frac{2\beta+1}{\beta-3}$ (b) $\frac{3\alpha+1}{\alpha-2}, \frac{3\beta+1}{\beta-2}$

(c) $\frac{2\alpha-1}{\alpha-2}, \frac{2\beta+1}{\beta-2}$ (d) none of these

4. Sum of series $\sum_{r=1}^{n} (r^2+1)\, r!$ is

(a) $(n + 1)!$ (b) $(n + 2)! - 1$

(c) $n(n + 1)!$ (d) none of these

5. If the sides a, b, c of a triangle ABC are in A.P., then $\frac{b}{c}$ belongs to

(a) $(0, 2/3)$ (b) $(1, 2)$

(c) $(2/3, 2)$ (d) $(2/3, 7/3)$

6. The sum of coefficients of the last eight terms in the expansion of $(1 + x)^{16}$ is equal to

(a) 2^{15} (b) 2^{14}

(c) $2^{15} - \frac{1}{2}\frac{(16)!}{(8!)^2}$ (d) none of these

7. Let $ax + by + c = 0$ be a variable straight line, where a, b and c are 1st, 3rd and 7th terms of an increasing A.P. respectively. Then the variable straight line always passes through a fixed point which lies on

(a) $y^2 = 4x$ (b) $x^2 + y^2 = 5$

(c) $3x + 4y = 9$ (d) $x^2 + y^2 = 13$

8. Three equal circles each of radius r touch one another. The radius of the circle touching all the three given circles internally is

(a) $(2+\sqrt{3})\,r$ (b) $\frac{(2+\sqrt{3})}{\sqrt{3}}\,r$

(c) $\frac{(2-\sqrt{3})}{\sqrt{3}}\,r$ (d) $(2-\sqrt{3})\,r$

9. The equation of the tangent to the parabola $y = (x - 3)^2$ parallel to the chord joining the points $(3, 0)$ and $(4, 1)$ is

(a) $2x - 2y + 6 = 0$ (b) $2y - 2x + 6 = 0$

(c) $4y - 4x + 13 = 0$ (d) $4x + 4y = 13$

10. The number of rational points on the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is

(a) ∞ (b) 4 (c) 0 (d) 2

11. The angle between the tangents from $(-2, -1)$ to the hyperbola $2x^2 - 3y^2 = 6$ is

(a) $\tan^{-1}(2)$ (b) $\pi/3$

(c) $\tan^{-1}(1/2)$ (d) $\pi/6$

12. If a function $F(x)$ satisfies the functional equation $x^2 F(x) + F(1-x) = 2x - x^4$ for all real x. $F(x)$ must be
(a) x^2 (b) $1 - x^2$
(c) $1 + x^2$ (d) $x^2 + x + 1$

13. The function $f(x) = [x]^2 - [x^2]$ (where $[y]$ is the greatest integer less than or equal to y), is discontinuous at
(a) all integers
(b) all integers except 0 and 1
(c) all integers except 0
(d) all integers except 1

14. The points of contact of the vertical tangents to the curve whose parametric equation is given as $x = 2 - 3\sin\theta$, $y = 3 + 2\cos\theta$ (where θ is a parameter) are
(a) (2, 5), (2, 1) (b) (–1, 3), (5, 3)
(c) (2, 5), (5, 3) (d) (–1, 3), (2, 1)

15. If $f(x)$ and $g(x) = f(x)\sqrt{1-2(f(x))^2}$ are monotonically increasing, then $\forall\ x \in R$
(a) $|f(x)| \le 1$ (b) $|f(x)| < \frac{2}{3}$
(c) $|f(x)| < \frac{1}{2}$ (d) $|f(x)| < \frac{1}{\sqrt{2}}$

16. Let $f(x) = \begin{cases} 2x^2 + 2/x^2 & ; \quad 0 < |x| \le 2 \\ 1 & ; \quad x = 0 \end{cases}$
Then $f(x)$ has
(a) least value 4 but no greatest value
(b) greatest value 4
(c) neither greatest nor least value
(d) least value 1 but no greatest value

17. If $f\left(\frac{3x-4}{3x+4}\right) = x+2$, then $\int f(x)dx$ is equal to
(a) $e^{x+2}\ln\left|\frac{3x-4}{3x+4}\right| + c$ (b) $-\frac{8}{3}\ln|(1-x)| + \frac{2}{3}x + c$
(c) $\frac{8}{3}\ln|x-1| + \frac{x}{3} + c$ (d) none of these

18. If A_n is the area bounded by $y = x$ and $y = x^n$, $n \in N$, then $A_2 . A_3 .... A_n =$
(a) $\frac{1}{n(n+1)}$ (b) $\frac{1}{2^n n(n+1)}$
(c) $\frac{1}{2^{n-1} n(n+1)}$ (d) $\frac{1}{2^{n-2} n(n+1)}$

19. If $|z^2 - 1| = |z|^2 + 1$, then z lies on a
(a) circle (b) parabola
(c) ellipse (d) none of these

20. In a quadrilateral $ABCD$, let

$$\Delta = \begin{vmatrix} \cos A & \sin A & \cos(A+D) \\ \cos B & \sin B & \cos(B+D) \\ \cos C & \sin C & \cos(C+D) \end{vmatrix}, \text{ then } \Delta \text{ is}$$

(a) independent of A and B only
(b) independent of B and C only
(c) independent of A, B and C only
(d) independent of A, B, C and D all

NUMERICAL VALUE TYPE

21. If $(1+x)^5 = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$. Then the value of $(a_0 - a_2 + a_4)^2 + (a_1 - a_3 + a_5)^2$ is equal to ______.

22. If all the words formed from the letters of the word "HORROR" are arranged in the opposite order as they are in a dictionary, then the rank of the word "HORROR" is ______.

23. If $\lim_{x\to 0} \frac{(2^{\sin x} - 1)[\ln(1+\sin 2x)]}{x\tan^{-1}x}$ is equal to k ln$k$, then $k =$ ______.

24. If $y = \frac{1}{x}$, then the value of $\frac{dy}{\sqrt{1+y^4}} + \frac{dx}{\sqrt{1+x^4}} + 3$ is equal to ______.

25. If $I_1 = \int_0^{\pi/2} \frac{x}{\sin x}dx$ and $I_2 = \int_0^1 \frac{\tan^{-1}x}{x}dx$, then $\frac{I_1}{I_2} =$ ______.

26. The degree of the differential equation whose general solution is given by

$$y = (c_1 + c_2)\cos(x + c_3) - c_4 e^{x+c_5}$$

where c_1, c_2, c_3, c_4, c_5 are arbitrary constants, is ______.

27. A can hit a target 4 times in 5 shots, B three times in 4 shots and C twice in 3 shots. They fire a target if exactly two of them hit the target, then the chance that it is C who has missed is ______.

28. If $A = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$, then $A + 2A^T = \lambda A^T$, where λ equals ______.

29. A vector of magnitude 3, bisecting the angle between the vectors $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$ and making an obtuse angle with $\vec{b}$ is $\dfrac{l(\hat{i} + l\hat{j} - m\hat{k})}{\sqrt{n}}$, where $l + m + n =$ ______.

30. If plane $ax - by + cz = d$ contains the line $\dfrac{x-a}{a} = \dfrac{y-2d}{b} = \dfrac{z-c}{c}$, then $\dfrac{b}{d} =$ ______.

SOLUTIONS

1. (d) : (a) $\cos(\tan^{-1}(\tan 4)) = \cos(\tan^{-1}\tan(4 - \pi))$
$= \cos(4 - \pi) = -\cos 4 > 0$

(b) $\sin(\cot^{-1}(\cot 4)) = \sin(\cot^{-1}\cot(4 - \pi))$
$= \sin(4 - \pi) = -\sin 4 > 0$

(c) $\tan(\cos^{-1}(\cos 5)) = \tan(\cos^{-1}\cos(2\pi - 5))$
$= \tan(2\pi - 5) = -\tan 5 > 0$

(d) $\cot(\sin^{-1}(\sin 4)) = \cot(\sin^{-1}\sin(\pi - 4))$
$= \cot(\pi - 4) = -\cot 4 < 0$

2. (a) : Since $4\sin A \cos B = 1$, so A and B cannot be 90° (as if $B = 90^\circ$, then $\cos B = 0$ and if $A = 90^\circ$, $\tan A$ is not defined)

$\therefore\ C = 90^\circ$ and $B = 90^\circ - A$

$\therefore\ 4\sin A \cos(90^\circ - A) = 1$

$\Rightarrow \sin^2 A = \dfrac{1}{4} \Rightarrow \sin A = \dfrac{1}{2} \Rightarrow A = \dfrac{\pi}{6} \Rightarrow B = \dfrac{\pi}{3}$

So angle $\dfrac{\pi}{6}, \dfrac{\pi}{3}, \dfrac{\pi}{2}$ are in A.P.

3. (b) : $a\dfrac{(2x+1)^2}{(x-3)^2} + b\dfrac{(2x+1)}{(x-3)} + c = 0$

$\Rightarrow \dfrac{2x+1}{x-3} = \alpha$ or $\dfrac{2x+1}{x-3} = \beta$

$\Rightarrow 2x + 1 = \alpha x - 3\alpha \Rightarrow x(\alpha - 2) = 1 + 3\alpha$

$\Rightarrow x = \dfrac{1+3\alpha}{\alpha-2}, \dfrac{1+3\beta}{\beta-2}$

4. (c) : $T_r = (r^2 + 1 + r - r)\, r!$

$\Rightarrow T_r = r(r+1)! - (r-1)r! \Rightarrow S_n = n(n+1)!$

5. (c) : $a = 2b - c$

$a + b > c \Rightarrow 2b - c + b > c \Rightarrow \dfrac{b}{c} > \dfrac{2}{3}$

Also, $b + c > a \Rightarrow b + c > 2b - c \Rightarrow \dfrac{b}{c} < 2$

Again $c + a > b \Rightarrow 2b > b \Rightarrow b > 0$

$\therefore\ \dfrac{b}{c} \in \left(\dfrac{2}{3}, 2\right)$

6. (c) : Sum of the coefficients of last eight terms

$$= {}^{16}C_9 + {}^{16}C_{10} + \ldots + {}^{16}C_{16} = \frac{2^{16} - {}^{16}C_8}{2}$$

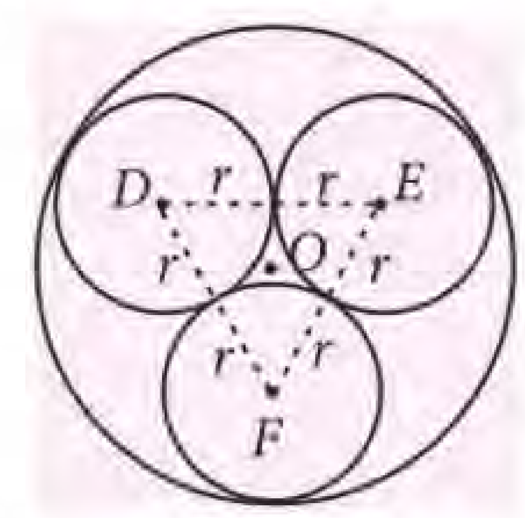
7. (d) : Let the common difference of A.P. is d then $b = a + 2d$ and $c = a + 6d$, so variable straight line will be

$ax + (a + 2d)y + a + 6d = 0$

$\Rightarrow a(x + y + 1) + d(2y + 6) = 0$,

which always passes through $(2, -3)$.

8. (b) : ΔDEF is equilateral with side $2r$. If radius of circumcircle DEF is R_1, then



Area of $\Delta DEF = \dfrac{\sqrt{3}}{4}(2r)^2 = \sqrt{3}r^2$

$$\sqrt{3}r^2 = \frac{2r \cdot 2r \cdot 2r}{4R_1} \Rightarrow R_1 = \frac{2r}{\sqrt{3}}$$

$\therefore$ Radius of the circle touching all the three given circles $= r + R_1$

$$= r + \frac{2r}{\sqrt{3}} = \frac{(2+\sqrt{3})r}{\sqrt{3}}$$

9. (c) : $y' = 2(x - 3) = 1$ gives the point $\left(\dfrac{7}{2}, \dfrac{1}{4}\right)$ and the required tangent is $y - \dfrac{1}{4} = 1\left(x - \dfrac{7}{2}\right)$

or $4y - 4x + 13 = 0$.

10. (a) : The given equation of ellipse is $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

Let any point on ellipse be $(3\cos\theta, 2\sin\theta)$

Since $\sin\theta$ and $\cos\theta$ can be rational for infinite many values of $\theta \in [0, 2\pi]$.

11. (c) : $\dfrac{x^2}{3} - \dfrac{y^2}{2} = 1;\ y + 1 = m(x + 2)$

or $y = mx + (2m - 1)$ touches the hyperbola.

SOLUTION SENDERS

Samurai Sudoku (March)

- Soumyakanti Mishra

Samurai Sudoku (April)

- Haridyal (Ahmedabad)

Mathdoku (April)

- Pratibha Sen (Gujarat)

$\because\ c^2 = a^2m^2 - b^2\ \therefore\ (2m-1)^2 = 3m^2 - 2$

$\Rightarrow m^2 - 4m + 3 = 0 \Rightarrow m = 1$ and 3

$\therefore\ \tan\theta = \left|\frac{3-1}{1+3\times1}\right| = \frac{1}{2} \Rightarrow \theta = \tan^{-1}\left(\frac{1}{2}\right)$

12. (b) : We have, $x^2F(x) + F(1-x) = 2x - x^4$ ...(i)
Replacing x by $(1-x)$ gives
$(1-x)^2 F(1-x) + F(x) = 2(1-x) - (1-x)^4$ ...(ii)
Eliminating $F(1-x)$ from (i) and (ii), we get
$F(x) = 1 - x^2$

13. (d) : Note that $f(x) = 0$ for each integral value of x.
Also, if $0 \le x < 1$, then $0 \le x^2 < 1$
$\therefore\ [x] = 0$ and $[x^2] = 0 \Rightarrow f(x) = 0$ for $0 \le x < 1$.
Next, if $1 \le x < \sqrt{2}$, then
$1 \le x^2 < 2 \Rightarrow [x] = 1$ and $[x^2] = 1$
Thus, $f(x) = [x]^2 - [x^2] = 0$ if $1 \le x < \sqrt{2}$
It follows that $f(x) = 0$, if $0 \le x < \sqrt{2}$
This shows that $f(x)$ must be continuous at $x = 1$.
However, at points x other than integers and not lying between 0 and $\sqrt{2}$, $f(x) \ne 0$.
Thus, f is discontinuous at all integers except 1.

14. (b) : For vertical tangents $\frac{dx}{d\theta} = 0$

so, we have $-3\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

Corresponding to these values of θ, we have

$x = 2 - 3\sin\frac{\pi}{2} = -1,\ y = 3 + 2\cos\frac{\pi}{2} = 3;$

$x = 2 - 3\sin\frac{3\pi}{2} = 2 + 3 = 5,\ y = 3 + 2\cos\frac{3\pi}{2} = 3$

Thus the required points are $(-1, 3)$, $(5, 3)$.

15. (c) : $g'(x) = \frac{[1 - 4(f(x))^2]f'(x)}{\sqrt{1 - 2(f(x))^2}}$

Now, as $f(x)$ and $g(x)$ are monotonically increasing, $f'(x) > 0$ and $g'(x) > 0 \Rightarrow |f(x)| < \frac{1}{2}$

16. (d) : For $x \to 0$

$2x^2 + \frac{2}{x^2} \to \infty$. Also $2\left(x^2 + \frac{1}{x^2}\right) \ge 4$

17. (b) : Put $\frac{3x-4}{3x+4} = t$

$\Rightarrow 3x - 4 = 3xt + 4t \Rightarrow x = \frac{4t+4}{3(1-t)}$

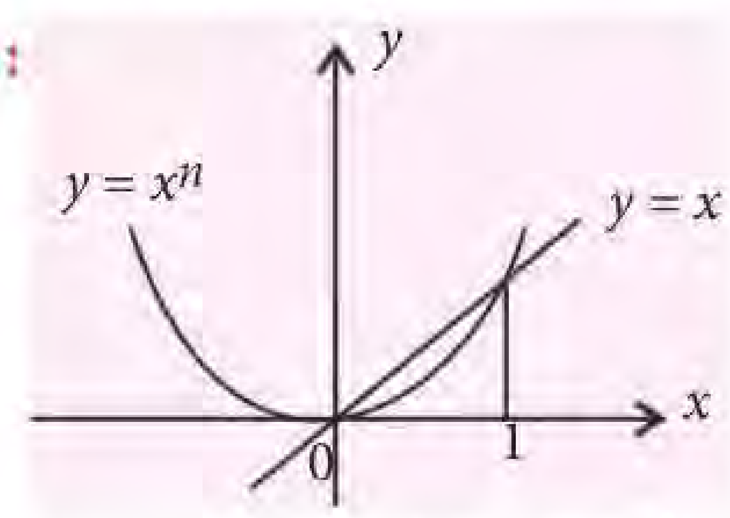
$f(t) = \frac{4t+4}{3(1-t)} + 2$

$\Rightarrow f(x) = \frac{4x+4}{3(1-x)} + 2 = \frac{4(x-1)+8}{3(1-x)} + 2$

$\Rightarrow f(x) = 2 - \frac{4}{3} - \frac{8}{3(x-1)} = \frac{2}{3} - \frac{8}{3(x-1)}$

$\therefore\ \int f(x)dx = \frac{2}{3}x - \frac{8}{3}\ln|x-1| + c$

18. (d) :



$$A_n = \int_0^1 (x - x^n)\,dx = \left[\frac{x^2}{2} - \frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{2} - \frac{1}{n+1} = \frac{n-1}{2(n+1)}$$

Thus $A_2 \cdot A_3 \cdot A_4 \ldots\ldots A_n = \frac{1}{2^{n-1}}\left(\frac{1}{3}\cdot\frac{2}{4}\cdot\frac{3}{5}\cdots\frac{n-1}{n+1}\right)$

$$= \frac{1}{2^{n-2}\cdot n(n+1)}$$

19. (d) : On putting $z = x + iy$ the equation is same as
$|x^2 - y^2 + 2ixy - 1| = x^2 + y^2 + 1$
$\Rightarrow (x^2 - y^2 - 1)^2 + 4x^2y^2 = (x^2 + y^2 + 1)^2 \Rightarrow x = 0$
$\Rightarrow z$ lies on imaginary axis, so (a), (b), (c) are ruled out.

20. (d) : Applying $C_3 \to C_3 - C_1\cos D + C_2\sin D$, we get $\Delta = 0$, hence Δ is independent of A, B, C, D all.

21. (32) : Put $x = i$
$(1+i)^5 = (a_0 - a_2 + a_4) + i\,(a_1 - a_3 + a_5)$
$\Rightarrow |1+i|^5 = |(a_0 - a_2 + a_4) + i\,(a_1 - a_3 + a_5)|$
$\Rightarrow (a_0 - a_2 + a_4)^2 + (a_1 - a_3 + a_5)^2 = 2^5 = 32$

22. (58) : RRROOH

R$\square\square\square\square\square$ $\frac{5!}{2!2!} = 30$

O$\square\square\square\square\square$ $\frac{5!}{3!} = 20$

HR$\square\square\square\square$ $\frac{4!}{2!2!} = 6$

HORRRO-1 HORROR-1

$\therefore$ Total = 30 + 20 + 6 + 1 + 1 = 58

23. (2) : $\lim_{x\to0}\frac{(2^{\sin x} - 1)[\ln(1 + \sin 2x)]}{\frac{x^2\tan^{-1}x}{x}}$

$= \lim_{x\to0}\frac{2^{\sin x} - 1}{\sin x} \times \frac{\sin x}{x} \times \frac{\ln(1 + \sin 2x)}{\sin 2x} \times \frac{\sin 2x}{2x} \times 2$

$= 2\ln 2 = k\ln k \Rightarrow k = 2$

24. (3) : $y=\frac{1}{x} \Rightarrow \frac{dy}{dx}=-\frac{1}{x^2} \Rightarrow x^2dy+dx=0$

$$\Rightarrow \frac{x^2}{\sqrt{1+x^4}}dy+\frac{dx}{\sqrt{1+x^4}}=0$$

$$\Rightarrow \frac{dy}{\sqrt{\frac{1}{x^4}+1}}+\frac{dx}{\sqrt{1+x^4}}=0 \Rightarrow \frac{dy}{\sqrt{1+y^4}}+\frac{dx}{\sqrt{1+x^4}}=0$$

$$\Rightarrow \frac{dy}{\sqrt{1+y^4}}+\frac{dx}{\sqrt{1+x^4}}+3=3$$

25. (2) : $I_2=\int_0^1 \frac{\tan^{-1}x}{x}dx, x=\tan\theta$

$$\Rightarrow I_2=\int_0^{\pi/4}\frac{2\theta}{\sin 2\theta}d\theta=\frac{1}{2}\int_0^{\pi/2}\frac{x}{\sin x}dx=\frac{1}{2}.I_1 \Rightarrow \frac{I_1}{I_2}=2$$

26. (1) : We can write $y = A\cos(x + B) - Ce^x$,

where $A = c_1 + c_2$, $B = c_3$ and $C = c_4e^{c_5}$

$$\frac{dy}{dx}=-A\sin(x+B)-Ce^x$$

$$\Rightarrow \frac{d^2y}{dx^2}=-A\cos(x+B)-Ce^x \Rightarrow \frac{d^2y}{dx^2}+y=-2Ce^x$$

$$\Rightarrow \frac{d^3y}{dx^3}+\frac{dy}{dx}=-2Ce^x=\frac{d^2y}{dx^2}+y$$

$$\Rightarrow \frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}+\frac{dy}{dx}-y=0,$$

which is a differential equation of degree 1.

27. (0.46) : Let A represents the event 'A hits the target', B represents the event 'B hits the target', C represents the event 'C hits the target' and E be the event that exactly two of A, B and C hit the target.

Then $P(A)=\frac{4}{5}$, $P(B)=\frac{3}{4}$ and $P(C)=\frac{2}{3}$

$\therefore\ P(C^c/E)$

$$=\frac{P(A)P(B)P(C^c)}{P(A)P(B)P(C^c)+P(A)P(B^c)P(C)+P(A^c)P(B)P(C)}$$

$$=\frac{6}{13}=0.461$$

28. (1) : $A^T=\begin{bmatrix}0 & 1 & -2\\ -1 & 0 & -3\\ 2 & 3 & 0\end{bmatrix}, 2A^T=\begin{bmatrix}0 & 2 & -4\\ -2 & 0 & -6\\ 4 & 6 & 0\end{bmatrix}$

Thus, $2A^T+A=\begin{bmatrix}0 & 1 & -2\\ -1 & 0 & -3\\ 2 & 3 & 0\end{bmatrix}=A^T$

29. (19) : A vector bisecting the angle between $\vec{a}$ and $\vec{b}$ is

$\frac{\vec{a}}{|\vec{a}|}\pm\frac{\vec{b}}{|\vec{b}|}$; in the case $\frac{2\hat{i}+\hat{j}-\hat{k}}{\sqrt{6}}\pm\frac{\hat{i}-2\hat{j}+\hat{k}}{\sqrt{6}}$

i.e., $\frac{3\hat{i}-\hat{j}}{\sqrt{6}}$ or $\frac{\hat{i}+3\hat{j}-2\hat{k}}{\sqrt{6}}$

A vector of magnitude 3 along these vectors is

$\frac{3(3\hat{i}-\hat{j})}{\sqrt{10}}$ or $\frac{3(\hat{i}+3\hat{j}-2\hat{k})}{\sqrt{14}}$

Now, $\frac{3}{\sqrt{14}}(\hat{i}+3\hat{j}-2\hat{k})\cdot(\hat{i}-2\hat{j}+\hat{k})$ is negative and hence

$\frac{3}{\sqrt{14}}(\hat{i}+3\hat{j}-2\hat{k})$ makes an obtuse angle with $\vec{b}$

30. (2) : Given plane contains the line

$\Rightarrow\ a^2 - b^2 + c^2 = 0$ ... (i)

and $a^2 - 2bd + c^2 = 0$ ...(ii)

By using (i) and (ii) we get $b/d = 2$

◈◈

SAMURAI SUDOKU

ANSWER - APRIL 2021

5	1	3	8	4	2	9	6	7				8	3	2	1	6	4	7	9	5
4	6	8	9	5	7	2	3	1				5	9	6	8	2	7	3	1	4
9	2	7	1	3	6	5	8	4				1	7	4	9	5	3	8	2	6
7	8	1	4	6	5	3	9	2				4	2	5	3	7	1	9	6	8
2	3	9	7	1	8	6	4	5				3	8	1	2	9	6	4	5	7
6	5	4	2	9	3	7	1	8				9	6	7	5	4	8	2	3	1
1	4	5	6	7	9	8	2	3	1	7	5	6	4	9	7	3	5	1	8	2
3	9	2	5	8	4	1	7	6	3	4	9	2	5	8	4	1	9	6	7	3
8	7	6	3	2	1	4	5	9	6	8	2	7	1	3	6	8	2	5	4	9
						7	6	1	4	3	8	5	9	2						
						2	4	5	7	9	6	3	8	1						
						3	9	8	2	5	1	4	6	7						
3	4	7	6	9	8	5	1	2	8	6	3	9	7	4	2	3	1	8	5	6
1	9	5	2	7	3	6	8	4	9	2	7	1	3	5	4	6	8	9	7	2
2	6	8	4	5	1	9	3	7	5	1	4	8	2	6	9	5	7	4	1	3
7	1	6	8	3	5	2	4	9				4	1	9	6	8	5	3	2	7
4	8	9	1	2	7	3	6	5				7	5	2	1	9	3	6	8	4
5	2	3	9	6	4	1	7	8				3	6	8	7	2	4	5	9	1
8	3	1	5	4	2	7	9	6				5	9	1	3	7	6	2	4	8
6	7	2	3	8	9	4	5	1				2	4	3	8	1	9	7	6	5
9	5	4	7	1	6	8	2	3				6	8	7	5	4	2	1	3	9

JEE ADVANCED PRACTICE PAPER

Exam on 3rd July 2021

PAPER-1

SINGLE OPTION CORRECT TYPE

1. Let p and q be real numbers such that $p \neq 0$, $p^3 \neq q$ and $p^3 \neq -q$. If α and β are non-zero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$, then a quadratic equation having $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ as its roots is

(a) $(p^3 + q)x^2 - (p^3 + 2q)x + (p^3 + q) = 0$
(b) $(p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$
(c) $(p^3 - q)x^2 - (5p^3 - 2q)x + (p^3 - q) = 0$
(d) $(p^3 - q)x^2 - (5p^3 + 2q)x + (p^3 - q) = 0$

2. The ellipse $E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point (0, 4) circumscribes the rectangle R. The eccentricity of the ellipse E_2 is

(a) $\frac{\sqrt{2}}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) $\frac{3}{4}$

3. Four fair dice D_1, D_2, D_3 and D_4, each having six faces numbered 1, 2, 3, 4, 5 and 6, are rolled simultaneously. The probability that D_4 shows a number appearing on one of D_1, D_2 and D_3 is

(a) $\frac{91}{216}$ (b) $\frac{108}{216}$ (c) $\frac{125}{216}$ (d) $\frac{127}{216}$

4. The area enclosed by the curves $y = \sin x + \cos x$ and $y = |\cos x - \sin x|$ over the interval $\left[0, \frac{\pi}{2}\right]$ is

(a) $4(\sqrt{2}-1)$ (b) $2\sqrt{2}(\sqrt{2}-1)$
(c) $2(\sqrt{2}+1)$ (d) $2\sqrt{2}(\sqrt{2}+1)$

5. Let $f : \left[\frac{1}{2}, 1\right] \to R$ (the set of all real numbers) be a positive, non-constant and differentiable function such that $f'(x) < 2f(x)$ and $f\left(\frac{1}{2}\right) = 1$. Then the value of $\int_{1/2}^{1} f(x)dx$ lies in the interval

(a) $(2e - 1, 2e)$ (b) $(e - 1, 2e - 1)$
(c) $\left(\frac{e-1}{2}, e-1\right)$ (d) $\left(0, \frac{e-1}{2}\right)$

6. Let f, g and h be real-valued functions defined on the interval [0, 1] by $f(x) = e^{x^2} + e^{-x^2}$, $g(x) = xe^{x^2} + e^{-x^2}$ and $h(x) = x^2e^{x^2} + e^{-x^2}$. If a, b and c denote, respectively, the absolute maximum of f, g and h on [0, 1], then

(a) $a = b$ and $c \neq b$ (b) $a = c$ and $a \neq b$
(c) $a \neq b$ and $c \neq b$ (d) $a = b = c$

ONE OR MORE THAN ONE OPTION(S) CORRECT TYPE

7. Let $f : [a, b] \to [1, \infty)$ be a continuous function and let $g : R \to R$ be defined as

$$g(x) = \begin{cases} 0 & , \text{ if } \quad x < a \\ \int_a^x f(t)dt, & \text{if } \quad a \leq x \leq b \\ \int_a^b f(t)dt, & \text{if } \quad x > b \end{cases}$$

Then

(a) $g(x)$ is continuous but not differentiable at a
(b) $g(x)$ is differentiable on R
(c) $g(x)$ is continuous but not differentiable at b
(d) $g(x)$ is continuous and differentiable at either a or b but not both

8. Let M and N be two 3 × 3 matrices such that $MN = NM$. Further, if $M \neq N^2$ and $M^2 = N^4$, then

(a) determinant of $(M^2 + MN^2)$ is 0
(b) there is a 3 × 3 non-zero matrix U such that $(M^2 + MN^2)U$ is the zero matrix

(c) determinant of $(M^2 + MN^2) \geq 1$
(d) for a 3×3 matrix U, if $(M^2 + MN^2)U$ equals the zero matrix then U is the zero matrix.

9. An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinate axes, then
(a) Equation of ellipse is $x^2 + 2y^2 = 2$
(b) The foci of ellipse are $(\pm 1, 0)$
(c) Equation of ellipse is $x^2 + 2y^2 = 4$
(d) The foci of ellipse are $(\pm\sqrt{2}, 0)$

10. If $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x}\, dx,\ n = 0, 1, 2, \ldots$, then
(a) $I_n = I_{n+2}$ (b) $\sum_{m=1}^{10} I_{2m+1} = 10\pi$
(c) $\sum_{m=1}^{10} I_{2m} = 0$ (d) $I_n = I_{n+1}$

11. Let $\vec{x}, \vec{y}$ and $\vec{z}$ be three vectors each of magnitude $\sqrt{2}$ and the angle between each pair of them is $\pi/3$. If $\vec{a}$ is a non-zero vector perpendicular to $\vec{x}$ and $\vec{y} \times \vec{z}$ and $\vec{b}$ is non-zero vector perpendicular to $\vec{y}$ and $\vec{z} \times \vec{x}$, then
(a) $\vec{b} = (\vec{b} \cdot \vec{z})(\vec{z} - \vec{x})$ (b) $\vec{a} = (\vec{a} \cdot \vec{y})(\vec{y} - \vec{z})$
(c) $\vec{a} \cdot \vec{b} = -(\vec{a} \cdot \vec{y})(\vec{b} \cdot \vec{z})$ (d) $\vec{a} = (\vec{a} \cdot \vec{y})(\vec{z} - \vec{y})$

12. Let z_1 and z_2 be two distinct complex numbers and let $z = (1 - t)z_1 + tz_2$ for some real number t with $0 < t < 1$. If $\text{Arg}(\omega)$ denotes the principal argument of a non-zero complex number ω, then
(a) $|z - z_1| + |z - z_2| = |z_1 - z_2|$
(b) $\text{Arg}(z - z_1) = \text{Arg}(z - z_2)$
(c) $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$
(d) $\text{Arg}(z - z_1) = \text{Arg}(z_2 - z_1)$

NUMERICAL VALUE TYPE

13. Suppose that $\vec{p}, \vec{q}$ and $\vec{r}$ are three non-coplanar vectors in R_3. Let the components of a vector $\vec{s}$ along $\vec{p}, \vec{q}$ and $\vec{r}$ be 4, 3 and 5 respectively. If the components of this vector $\vec{s}$ along $(-\vec{p}+\vec{q}+\vec{r}), (\vec{p}-\vec{q}+\vec{r})$ and $(-\vec{p}-\vec{q}+\vec{r})$ are x, y and z respectively, then the value of $2x + y + z$ is ______.

14. Let m and n be two positive integers greater than 1. If $\lim_{\alpha \to 0}\left(\frac{e^{\cos(\alpha^n)} - e}{\alpha^m}\right) = -\left(\frac{e}{2}\right)$, then the value of $\frac{m}{n}$ is ______.

15. Let X be the set consisting of the first 2018 terms of the arithmetic progression 1, 6, 11, ... , and Y be the set consisting of the first 2018 terms of the arithmetic progression 9, 16, 23, ... Then, the number of elements in the set $X \cup Y$ is ______.

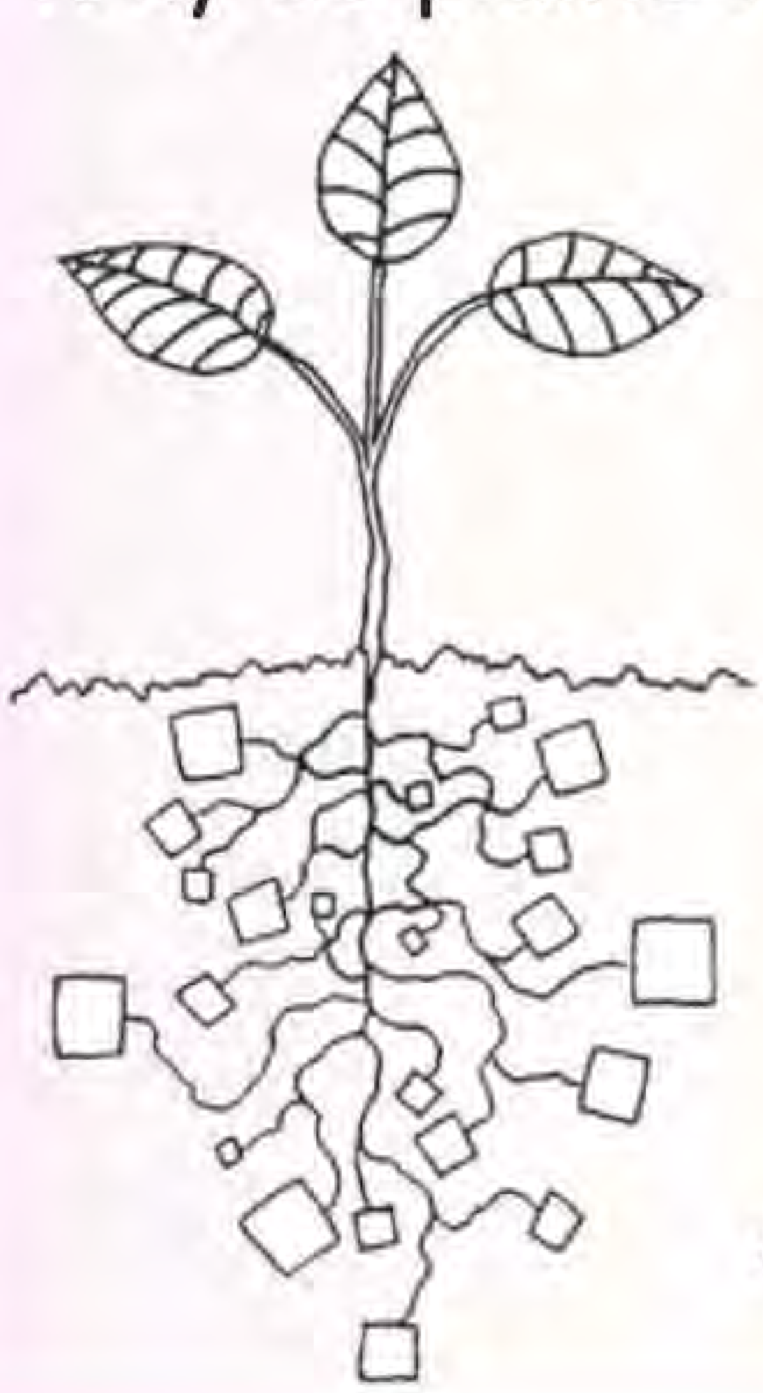
16. A farmer F_1 has a land in the shape of a triangle with vertices at $P(0, 0)$, $Q(1, 1)$ and $R(2, 0)$. From this land, a neighbouring farmer F_2 takes away the region which lies between the side PQ and a curve of the form $y = x^n$ $(n > 1)$. If the area of the region taken away by the farmer F_2 is exactly 30% of the area of ΔPQR, then the value of n is ______ .

17. A debate club consists of 6 girls and 4 boys. A team of 4 members is to be selected from this club including the selection of a captain (from among these 4 members) for the team. If the team has to include at most one boy, then the number of ways of selecting the team is ______.

18. Let $f : R \to R$ and $g : R \to R$ be respectively given by $f(x) = |x| + 1$ and $g(x) = x^2 + 1$. Define $h : R \to R$ by
$$h(x) = \begin{cases} \max\{f(x), g(x)\}, & \text{if } x \leq 0 \\ \min\{f(x), g(x)\}, & \text{if } x > 0 \end{cases}.$$
The number of points at which $h(x)$ is not differentiable is ______.

COMIC CAPSULE

Why do plants hate maths?



Because it gives them *square roots.*

PAPER-2

SINGLE DIGIT INTEGER ANSWER TYPE

1. If $y = y(x)$ satisfies the differential equation

$8\sqrt{x}(\sqrt{9+\sqrt{x}})dy = (\sqrt{4+\sqrt{9+\sqrt{x}}})^{-1}dx, x > 0$

and $y(0) = \sqrt{7}$, then $y(256) =$ ______.

2. If $\lim_{x\to 0} \frac{\sin(3x+a) - 3\sin(2x+a) + 3\sin(x+a) - \sin a}{x^3}$ $= -\cos 1$, then $a =$ ______.

3. Let k be a positive real number and

$$A = \begin{bmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix}, B = \begin{bmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{bmatrix}$$

If det (adj A) + det (adj B) = 10^6, then $[k]$ is ______.

4. Let $A = \{1, 2, 3, 4\}$ and $B = \{0, 1, 2, 3, 4, 5\}$. If m is the number of increasing functions from A to B and n is the number of onto functions from B to A, then the unit digit of $n - m$ is ______.

5. Let $f(x + y) = f(x) + f(y) - 2xy - 1$ for all x and y. If $f'(0)$ exists and $f'(0) = -\sin\alpha$, then the value of $f\{f'(0)\}$ is ______.

6. If $\omega \neq 1, \omega^3 = 1$, then the number of prime factors of $N = \sum_{r=1}^{10}(r-\omega)(r-\omega^2)$ is ______.

ONE OR MORE THAN ONE OPTION(S) CORRECT TYPE

7. If OT and ON are respectively the lengths of perpendiculars drawn from the origin to the tangent and normal drawn at any arbitrary point on the curve $x = a\sin^3 t, y = a\cos^3 t$, then

(a) $4OT^2 + ON^2 = a^2$

(b) the length of the tangent = $\left|\frac{y}{\cos t}\right|$

(c) the length of the normal = $\left|\frac{y}{\sin t}\right|$

(d) none of these

8. If the hyperbola $xy = c^2$ intersects the circle $x^2 + y^2 = a^2$ at four points $P_i(x_i, y_i)$, $i = 1, 2, 3, 4$, then

(a) $x_1 + x_2 + x_3 + x_4 = 0$

(b) $y_1 + y_2 + y_3 + y_4 = 0$

(c) $x_1x_2x_3x_4 = c^4$

(d) $y_1y_2y_3y_4 = c^4$

9. A man sent 7 letters to his 7 friends. The letters are kept in the envelopes at random. The number of ways of exactly 3 letters going to the correct destinations and 4 letters going to the wrong destinations is

(a) 210 (b) 315

(c) $5 \times 7 \times 3^2$ (d) 420

10. Let $\vec{A}$ be vector parallel to the line of intersection of planes P_1 and P_2 through the origin. P_1 is parallel to the vectors $\vec{a} = 2\hat{j} + 3\hat{k}$ and $\vec{b} = 4\hat{j} - 3\hat{k}$ and P_2 is parallel to the vectors $\vec{c} = \hat{j} - \hat{k}$ and $\vec{d} = 3\hat{i} + 3\hat{j}$. The angle between A and $2\hat{i} + \hat{j} - 2\hat{k}$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{3\pi}{4}$

11. Let $y = f(x)$ be a curve in the first quadrant such that the triangle formed by the co-ordinate axes and the tangent at any point on the curve has area 2. If $y(1) = 1$, then $y(2) =$

(a) 0 (b) 1 (c) 2 (d) $\frac{1}{2}$

12. The direction cosines of two lines are connected by relations $l + m + n = 0$ and $4l$ is the harmonic mean between m and n. Then,

(a) $\frac{l_1}{l_2} + \frac{m_1}{m_2} + \frac{n_1}{n_2} = -3/2$

(b) $l_1l_2 + m_1m_2 + n_1n_2 = -1/2$

(c) $l_1m_1n_1 + l_2m_2n_2 = -\sqrt{6}/9$

(d) $(l_1 + l_2)(m_1 + m_2)(n_1 + n_2) = \frac{\sqrt{6}}{18}$

NUMERICAL VALUE TYPE

13. A pack contains n cards numbered from 1 to n. Two consecutive numbered cards are removed from the pack and the sum of the numbers on the remaining cards is 1224. If the smaller of the numbers on the removed cards is k, then $k - 20$ equals ______.

14. Of the three independent events, E_1, E_2 and E_3 the probability that only E_1 occurs is α, only E_2 occurs is β and only E_3 occurs is γ. Let the probability p that none of the events E_1, E_2 or E_3 occurs satisfy the equations $(\alpha - 2\beta)p = \alpha\beta$ and $(\beta - 3\gamma)p = 2\beta\gamma$. All the given probabilities are assumed to lie in the interval (0, 1). Then $\frac{\text{Probability of occurrence of } E_1}{\text{Probability of occurrence of } E_3} =$ ______.

15. The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse $x^2 + 9y^2 = 9$ meets the auxiliary circle at the point M. The area of the triangle AMO is ______.

16. A man X has 7 friends, 4 of them are ladies and 3 are men. His wife Y also has 7 friends, 3 of them are ladies and 4 are men. Assume X and Y have no common friends. Then the total number of ways in which X and Y together can throw a party inviting 3 ladies and 3 men, so that 3 friends of each of X and Y are in this party, is ______.

17. Three randomly chosen non-negative integers x, y and z are found to satisfy the equation $x + y + z = 10$. Then the probability that z is even, is $6/\lambda$, where λ equals ______.

18. Let $f(x)$ be the ratio of two quadratic polynomials. If $f(0) = 6$ and $f(x)$ assumes turning values 3 and 4 at $x = 2$ and $x = -2$ respectively, then $f(1) =$ ______.

SOLUTIONS

PAPER-1

1. (b) : $\alpha + \beta = -p$

$\alpha^3 + \beta^3 = q$

They yields $(\alpha + \beta)^3 = \alpha^3 + \beta^3 + 3\alpha\beta(\alpha + \beta)$

$$\Rightarrow\ -p^3 = q - 3\alpha\beta p \quad \Rightarrow \alpha\beta = \frac{p^3+q}{3p}$$

Now $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\frac{(\alpha+\beta)^2}{\alpha\beta}-2$

$$=\frac{(p^2)(3p)}{p^3+q}-2=\frac{3p^3-2p^3-2q}{p^3+q}=\frac{p^3-2q}{p^3+q}$$

$\therefore$ Required equation is $x^2-\left(\frac{p^3-2q}{p^3+q}\right)x+1=0$

i.e., $(p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$

2. (c) : Let the required ellipse be $\frac{x^2}{\alpha}+\frac{y^2}{\beta}=1$

It passes through (0, 4) $\Rightarrow \frac{16}{\beta}=1 \ \therefore\ \beta=16$

It passes through (3, 2) $\Rightarrow \frac{9}{\alpha}+\frac{4}{\beta}=1 \Rightarrow \alpha=12$

Also $\alpha = \beta(1 - e^2) \Rightarrow 12 = 16(1 - e^2)$

$$\Rightarrow \frac{3}{4}=1-e^2 \Rightarrow e^2=\frac{1}{4} \ \therefore\ e=\frac{1}{2}$$

3. (a) : The number of ways in which one of D_1, D_2 and D_3 shows the same number as D_4 = the number of total ways – number of ways in which D_1, D_2, D_3 don't show a number appearing on $D_4 = 6^4 - {}^6C_1 \cdot 5^3$

$\therefore$ The required probability

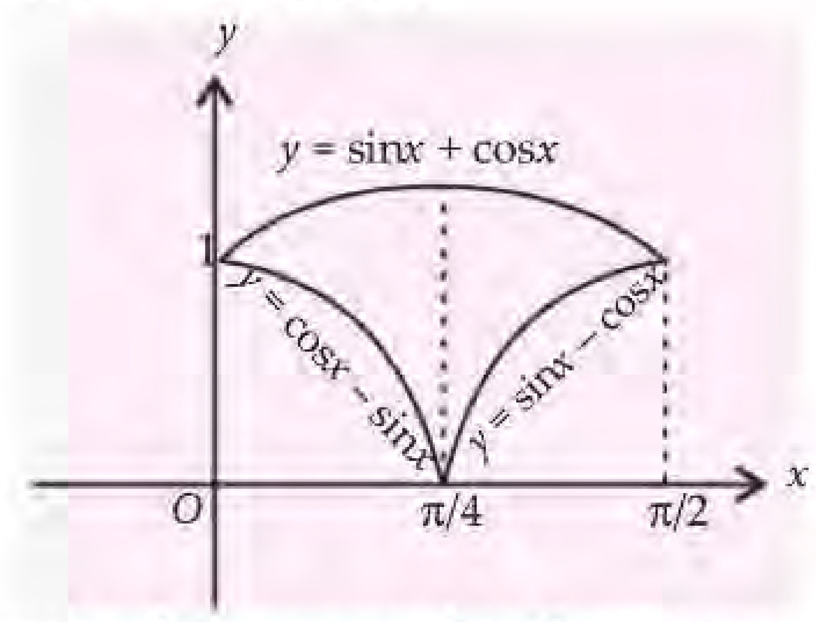
$$=\frac{6^4-6\cdot5^3}{6^4}=1-\left(\frac{5}{6}\right)^3=\frac{91}{216}$$

4. (b) : The curves are

$y = \sin x + \cos x,\ x \in [0, \pi/2]$

$$y=\begin{cases}\cos x-\sin x, & x\in[0,\pi/4)\\ \sin x-\cos x, & x\in(\pi/4,\pi/2]\end{cases}$$

A rough sketch is as under



The area required is set up by the integral

$$\int_0^{\pi/4}\{(\sin x+\cos x)-(\cos x-\sin x)\}\,dx + \int_{\pi/4}^{\pi/2}\{(\sin x+\cos x)-(\sin x-\cos x)\}dx$$

$$=2\int_0^{\pi/4}\sin x\,dx+2\int_{\pi/4}^{\pi/2}\cos x\,dx=2(-\cos x)\Big|_0^{\pi/4}+2(\sin x)\Big|_{\pi/4}^{\pi/2}$$

$$=2\left(1-\frac{1}{\sqrt2}\right)+2\left(1-\frac{1}{\sqrt2}\right)=4\left(1-\frac{1}{\sqrt2}\right)=2\sqrt2\,(\sqrt2-1)$$

5. (d) : Rewrite the given equation as

$f'(x) < 2f(x) \quad \Rightarrow f'(x) - 2f(x) < 0$

$$\Rightarrow e^{-2x}(f'(x) - 2f(x)) < 0 \quad \Rightarrow \frac{d}{dx}(e^{-2x}f(x))<0$$

Put $g(x) = e^{-2x}f(x)$. The condition thus reach $g'(x) < 0$.

Thus g is decreasing on $\left[\frac{1}{2}, 1\right]$.

Hence, $x>\frac{1}{2} \Rightarrow g(x)<g\left(\frac{1}{2}\right)$

$$\Rightarrow e^{-2x}f(x)<e^{-1}f\left(\frac{1}{2}\right) \quad \left[\because f\left(\frac{1}{2}\right)=1\right]$$

$$\Rightarrow f(x)<e^{2x-1}$$

On integrating both sides,

$$\int_{1/2}^1 f(x)\,dx<\int_{1/2}^1 e^{2x-1}\,dx=\left[\frac{e^{2x-1}}{2}\right]_{1/2}^1=\frac{1}{2}(e-1)$$

As f is positive, we have $\int_{1/2}^1 f(x)dx>0$

Hence, the interval in which the integral lies is $\left(0,\frac{e-1}{2}\right)$.

6. **(d) :** $f(x) = e^{x^2} + e^{-x^2}$

$\Rightarrow\ f'(x) = 2x(e^{x^2} - e^{-x^2}) \geq 0\ \forall x \in [0, 1]$

Thus f is increasing and so the maximum value of f is $f(1) = e + \frac{1}{e}$

Again $g(x) = xe^{x^2} + e^{-x^2}$

$\Rightarrow\ g'(x) = (1 + 2x^2)\, e^{x^2} - 2xe^{-x^2} \geq 0\ \forall x \in [0, 1]$

Thus g is increasing and so $g(1) = e + \frac{1}{e}$ is the maximum value.

Also $h(x) = x^2e^{x^2} + e^{-x^2}$

$\Rightarrow\ h'(x) = 2x[e^{x^2} + x^2e^{x^2} - e^{-x^2}] \geq 0\ \forall x \in [0, 1]$

Thus h is increasing and so $h(1) = e + \frac{1}{e}$ is the maximum value. Hence $a = b = c$.

7. **(a, c) :** $g(a^-) = 0$

Also, $g(a^+) = \lim_{h\to 0} \int_a^{a+h} f(t)dt = 0$

Thus g is continuous at $x = a$. Similarly, g is continuous at $x = b$.

$g'(a^-) = \lim_{h\to 0} \frac{g(a-h) - g(a)}{-h}$ (by definition)

$$= \lim_{h\to 0} \frac{0 - \int_a^a f(t)dt}{-h} = \lim_{h\to 0} \frac{0}{-h} = 0$$

$$g'(a^+) = \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} = \lim_{h\to 0} \frac{\int_a^{a+h} f(t)dt - 0}{h}$$

$$= \lim_{h\to 0} \frac{\int_a^{a+h} f(t)dt}{h} = \lim_{h\to 0} \frac{f(a+h)}{1} = f(a) \geq 1$$

So, $g'_{(a^-)} \neq g'_{(a^+)}$

Hence g is not differentiable at $x = a$.

Similarly, g is not differentiable at $x = b$.

8. **(a, b) :** $M^2 = N^4 \Rightarrow (M - N^2)(M + N^2) = 0$ as M and N commute.

$M - N^2 \neq 0$ so $\det(M + N^2) = 0$

Recall that if $AB = O$ and $A \neq 0$, then $\det B = 0$

Now, $\det(M^2 + MN^2) = \det(M(M + N^2))$

$= (\det M)(\det(M + N^2))$

$= (\det M) \cdot 0 = 0$

Recall that if $\det A = 0$, then $\exists\ X \neq 0$ such that $AX = 0$

Let $M^2 + MN^2$ play the role of A and X that of U, we get $(M^2 + MN^2)U = O$ for some 3×3 non-zero matrix U.

9. **(a, b) :** The hyperbola is $\frac{x^2}{1/2} - \frac{y^2}{1/2} = 1$ ...(i)

Eccentricity of the hyperbola $= \sqrt{2}$

$\therefore$ Eccentricity of the ellipse $= \frac{1}{\sqrt{2}}$

Let the equation of ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ (a > b)$

Then $a^2e^2 = a^2 - b^2$

$\Rightarrow a^2 \cdot \frac{1}{2} = a^2 - b^2 \Rightarrow b^2 = \frac{a^2}{2}$ $\quad \therefore a^2 = 2b^2$

The equation is $\frac{x^2}{2b^2} + \frac{y^2}{b^2} = 1$

$x^2 + 2y^2 = 2b^2$ .....(ii)

Let (x_0, y_0) be a point of intersection of hyperbola (i) and ellipse (ii),

Then for hyperbola $\left[\frac{dy}{dx}\right]_{(x_0,\, y_0)} = \frac{x_0}{y_0}$

and for ellipse $\left[\frac{dy}{dx}\right]_{(x_0,\, y_0)} = -\frac{x_0}{2y_0}$

As the intersection is orthogonal, we have

$-\frac{x_0^2}{2y_0^2} = -1 \Rightarrow x_0^2 = 2y_0^2$

From (i) and (ii), $2x_0^2 - 2y_0^2 = 1$ and $x_0^2 + 2y_0^2 = 2b^2$

Multiplying the 1st equation by $2b^2$ and subtracting we get $4b^2x_0^2 - 4y_0^2b^2 = x_0^2 + 2y_0^2$

$\Rightarrow\ x_0^2(4b^2 - 1) = 2y_0^2(1 + 2b^2)$

Also $x_0^2 = 2y_0^2$ which gives

$4b^2 - 1 = 1 + 2b^2 \Rightarrow 2b^2 = 2$

$\therefore\ b^2 = 1$

Thus the equation to ellipse is $x^2 + 2y^2 = 2$

$\therefore$ Foci of the ellipse are $\left(\pm\sqrt{a^2 - b^2}, 0\right)$ *i.e.* $(\pm 1, 0)$

10. **(a, b, c) :** $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x}dx$ ...(i)

$\Rightarrow\ I_n = \int_{-\pi}^{\pi} \frac{\sin(-nx)}{(1+\pi^{-x})(-\sin x)}dx$ ...(ii)

Adding (i) and (ii), we get

$$2I_n = \int_{-\pi}^{\pi} \left(\frac{1}{1+\pi^x} + \frac{1}{1+\pi^{-x}}\right)\frac{\sin nx}{\sin x}dx = \int_{-\pi}^{\pi} \frac{\sin nx}{\sin x}dx$$

$$\Rightarrow\ I_n = \frac{1}{2}\int_{-\pi}^{\pi} \frac{\sin nx}{\sin x}dx$$

Now, $I_{n+2} - I_n = \frac{1}{2}\int_{-\pi}^{\pi} \frac{\sin(n+2)x - \sin nx}{\sin x}dx$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \frac{2\sin x\cos(n+1)x}{\sin x}dx$$

$$= \int_{-\pi}^{\pi} \cos(n+1)x\,dx = \left(\frac{\sin(n+1)x}{(n+1)}\right)_{-\pi}^{\pi} = 0$$

$\therefore \quad I_{n+2} = I_n$

Thus $I_0 = I_2 = I_4 = \ldots\ldots = I_{10}$

$I_1 = I_3 = I_5 = \ldots\ldots = I_{21}$

$$I_0 = \frac{1}{2}\int_{-\pi}^{\pi} 0\,dx = 0$$

$$\therefore \quad I_1 = \frac{1}{2}\int_{-\pi}^{\pi} \frac{\sin x}{\sin x}dx = \frac{1}{2}\cdot 2\pi = \pi$$

$$\sum_{m=1}^{10} I_{2m} = 0 \text{ and } \sum_{m=1}^{10} I_{2m+1} = 10\pi$$

11. (a, b, c) : $\vec{a} = p(\vec{x}\times(\vec{y}\times\vec{z}))$, where $p \in R - \{0\}$

$= p\{(\vec{x}\cdot\vec{z})\vec{y} - (\vec{x}\cdot\vec{y})\vec{z}\} = p\{\vec{y} - \vec{z}\}$

$$\left(\because \vec{x}\cdot\vec{y} = |x||y|\cos\frac{\pi}{3} = \sqrt{2}.\sqrt{2}\left(\frac{1}{2}\right) = 1 = \vec{y}\cdot\vec{z} = \vec{z}\cdot\vec{x}\right)$$

Now, $\vec{a}\cdot\vec{y} = p\{\vec{y}\cdot\vec{y} - \vec{y}\cdot\vec{z}\} = p\{2-1\} = p$

$\therefore \quad p = \vec{a}\cdot\vec{y} \qquad (\because \vec{y}\cdot\vec{y} = \vec{y}^2 = (\sqrt{2})^2 = 2)$

Thus, $\vec{a} = (\vec{a}\cdot\vec{y})(\vec{y} - \vec{z})$ ..... (i)

Similarly, $\vec{b} = q(\vec{y}\times(\vec{z}\times\vec{x})) = q\{(\vec{y}\cdot\vec{x})\vec{z} - (\vec{y}\cdot\vec{z})\vec{x}\}$

$= q(\vec{z} - \vec{x})$

Now, $\vec{b}\cdot\vec{z} = q(\vec{z}\cdot\vec{z} - \vec{z}\cdot\vec{x}) = q\,(\vec{z}^2 - \vec{z}\cdot\vec{x}) = q$

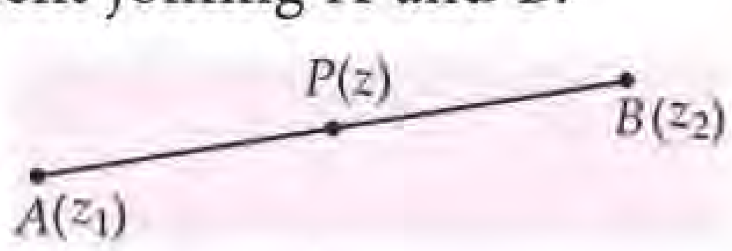
Thus $\vec{b} = (\vec{b}\cdot\vec{z})(\vec{z} - \vec{x})$ .....(ii)

Now, $\vec{a}\cdot\vec{b} = (\vec{a}\cdot\vec{y})(\vec{b}\cdot\vec{z})\{(\vec{y}-\vec{z})\cdot(\vec{z}-\vec{x})\}$ (From (i) & (ii))

$= (\vec{a}\cdot\vec{y})(\vec{b}\cdot\vec{z})(\vec{y}\cdot\vec{z} - \vec{y}\cdot\vec{x} - \vec{z}\cdot\vec{z} + \vec{z}\cdot\vec{x})$

$= (\vec{a}\cdot\vec{y})(\vec{b}\cdot\vec{z})\,(1-1-2+1) = -(\vec{a}\cdot\vec{y})(\vec{b}\cdot\vec{z})$

12. (a, c, d) : The given statement implies that z is on the line segment joining A and B.



As $z = (1-t)z_1 + tz_2$ indicate.

$\therefore \quad AP + PB = AB \Rightarrow |z - z_1| + |z - z_2| = |z_1 - z_2|$

Now, z_1, z, z_2 are collinear, so $\operatorname{Arg}\left(\frac{z - z_1}{z_2 - z_1}\right) = 0$

giving $\operatorname{Arg}(z - z_1) = \operatorname{Arg}(z_2 - z_1)$

Note that $\operatorname{Arg}(z - z_1) - \operatorname{Arg}(z - z_2) = \pi$

Also $\frac{z - z_1}{\bar{z} - \bar{z}_1} = \frac{z_2 - z_1}{\bar{z}_2 - \bar{z}_1}$ = complex slope of the line which can be rearranged as

$(z - z_1)(\bar{z}_2 - \bar{z}_1) - (z_2 - z_1)(\bar{z} - \bar{z}_1) = 0$

i.e., $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$

13. (9) : Given, $\vec{s} = 4\vec{p} + 3\vec{q} + 5\vec{r}$

Let $-\vec{p} + \vec{q} + \vec{r} = \vec{\alpha}$, $\vec{p} - \vec{q} + \vec{r} = \vec{\beta}$ and $-\vec{p} - \vec{q} + \vec{r} = \vec{\gamma}$

Then solving for $\vec{p}, \vec{q}$ and $\vec{r}$, we have

$$\vec{p} = \frac{\vec{\beta} - \vec{\gamma}}{2},\ \vec{q} = \frac{\vec{\alpha} - \vec{\gamma}}{2},\ \vec{r} = \frac{\vec{\alpha} + \vec{\beta}}{2}$$

Rewrite, $\vec{s} = 4\left(\frac{\vec{\beta} - \vec{\gamma}}{2}\right) + 3\left(\frac{\vec{\alpha} - \vec{\gamma}}{2}\right) + 5\left(\frac{\vec{\alpha} + \vec{\beta}}{2}\right)$

$$= 4\vec{\alpha} + \frac{9}{2}\vec{\beta} - \frac{7}{2}\vec{\gamma}$$

$\therefore \quad x = 4,\ y = \frac{9}{2},\ z = \frac{-7}{2}$

Thus, $2x + y + z = 9$

14. (2) : $\lim_{\alpha\to 0}\left(\frac{e^{\cos(\alpha^n)} - e}{\alpha^m}\right) = \lim_{\alpha\to 0}\frac{e(e^{\cos(\alpha^n)-1} - 1)}{\alpha^m}$

$$= \lim_{\alpha\to 0}\frac{e(\alpha^n)^2}{\alpha^m}\left\{\frac{e^{\cos(\alpha^n)-1} - 1}{\cos(\alpha^n) - 1}\right\}\cdot\left\{\frac{\cos(\alpha^n) - 1}{(\alpha^n)^2}\right\} = \frac{-1}{2}e\alpha^{2n-m}$$

As limit is given to be $-e/2$, we have

$2n - m = 0. \quad \therefore \ m/n = 2.$

15. (3748) : Here X = {1, 6, 11, ..., 10086}

and Y = {9, 16, 23, ..., 14128}

The intersection of X and Y is an A.P. with 16 as first term and 35 as common difference.

The series becomes 16, 51, 86, ...

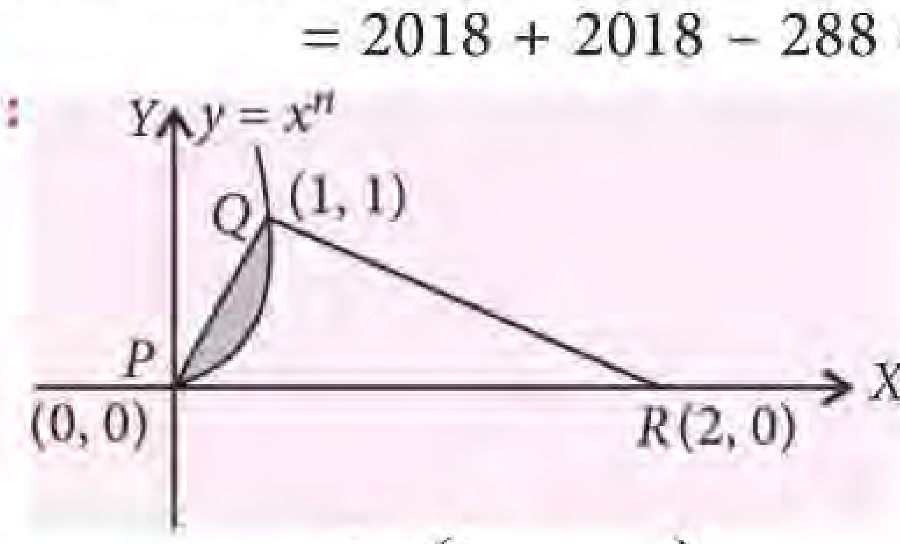
Now, k^{th} term $= 16 + (k-1)\,35 \le 10086$

i.e. $k \le \frac{10105}{35}$ $\therefore\ k \le 288$ (as k is to be an integer)

Hence, $n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$

$= 2018 + 2018 - 288 = 3748$

16. (4) :



$$\int_0^1 (x - x^n)dx = \frac{3}{10}\cdot\left(\frac{1}{2}\times 2\times 1\right)$$

$$\Rightarrow \left[\frac{x^2}{2} - \frac{x^{n+1}}{n+1}\right]_0^1 = \frac{3}{10} \quad \Rightarrow \frac{1}{2} - \frac{1}{n+1} = \frac{3}{10}$$

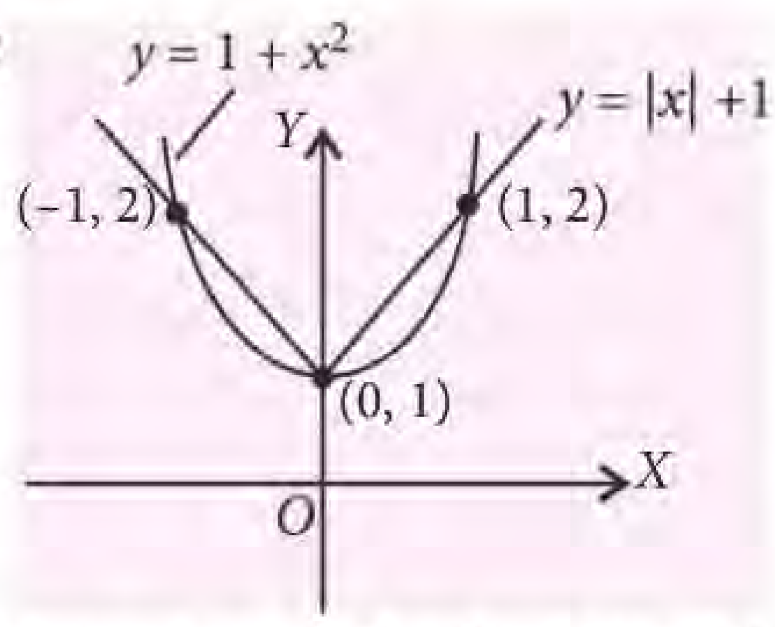
$$\Rightarrow \frac{1}{2} - \frac{3}{10} = \frac{1}{n+1} \quad \Rightarrow \frac{2}{10} = \frac{1}{n+1}$$

$\Rightarrow \ n + 1 = 5 \quad \therefore\ n = 4$

17. (380) : The team can have either 0 boy or 1 boy. So the number of selection is $({}^6C_4\cdot{}^4C_0 + {}^6C_3\cdot{}^4C_1)\cdot{}^4C_1$

$= (15 + 20\cdot 4)\cdot 4 = 95\cdot 4 = 380$

18. (3) :



From the given graph the function $h(x)$ in closed form is

$$h(x)=\begin{cases} x^2+1, & -\infty < x \le -1 \\ -x+1, & -1 < x \le 0 \\ x^2+1, & 0 < x \le 1 \\ x+1, & 1 < x < \infty \end{cases}$$

Hence the points of non-differentiability are –1, 0 and 1.

PAPER-2

1. (3) : Rewrite the differential equation as

$$\frac{dy}{dx}=\frac{1}{8\sqrt{x}\sqrt{9+\sqrt{x}}}\cdot\frac{1}{\sqrt{4+\sqrt{9+\sqrt{x}}}}$$

Let $P=\sqrt{4+\sqrt{9+\sqrt{x}}}$

Then, $\dfrac{dP}{dx}=\dfrac{1}{2\sqrt{4+\sqrt{9+\sqrt{x}}}}\cdot\dfrac{1}{2\sqrt{9+\sqrt{x}}}\cdot\dfrac{1}{2\sqrt{x}}$

Now, we have $\dfrac{dy}{dx}=\dfrac{dP}{dx}$. Then $y = P(x) + \lambda$

$\Rightarrow \quad y=\sqrt{4+\sqrt{9+\sqrt{x}}}+\lambda.$

Now, $y(0) = \sqrt{7}$ (Given) $\Rightarrow \lambda = 0$

Then, $y=\sqrt{4+\sqrt{9+\sqrt{x}}}$

$\therefore$ At $x = 256$, $y=\sqrt{4+\sqrt{9+\sqrt{256}}}=\sqrt{4+\sqrt{9+16}}=3$

2. (1) : $\displaystyle \operatorname*{Lt}_{x\to 0}\frac{\sin(3x+a)-3\sin(2x+a)+3\sin(x+a)-\sin a}{x^3}$

$$=\operatorname*{Lt}_{x\to 0}\frac{2\sin\frac{3x}{2}\cos\left(\frac{3x+2a}{2}\right)-3\left(2\sin\frac{x}{2}\cos\frac{3x+2a}{2}\right)}{x^3}$$

$$=\operatorname*{Lt}_{x\to 0}\frac{2\cos\left(\frac{3x+2a}{2}\right)\left[3\sin\frac{x}{2}-4\sin^3\frac{x}{2}-3\sin\frac{x}{2}\right]}{x^3}$$

$$=\operatorname*{Lt}_{x\to 0} 2\cos\left(\frac{3x+2a}{2}\right)\left\{\frac{-4}{8}\right\}\quad\left[\because \operatorname*{Lt}_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right]$$

$=2\cos a\left(-\dfrac{4}{8}\right) = -\cos a = -\cos 1 \quad \therefore a = 1$

3. (4) : $\det A = (2k-1)(4k^2-1) + 2\sqrt{k}\,(4k\sqrt{k}+2\sqrt{k})+2\sqrt{k}\,(4k\sqrt{k}+2\sqrt{k})$

$= (2k-1)(4k^2-1) + 8(2k+1)k$

$= (2k+1)((2k-1)^2+8k) = (2k+1)^3$

$\det(\operatorname{adj} A) = (\det A)^2 = (2k+1)^6$

$\det B = 0$ since B is skew symmetric matrix of order 3.

$\det(\operatorname{adj} B) = (\det B)^2 = 0 \quad \therefore \quad (2k+1)^6 = 10^6$

$\Rightarrow \quad 2k+1 = 10 \Rightarrow k = 4.5$

So, $[k] = [4.5] = 4$

4. (5) : $m=\dbinom{6}{4}=\dbinom{6}{2}=15$

$n=4^6-\dbinom{4}{1}3^6+\dbinom{4}{2}2^6-\dbinom{4}{3}1^6=1560$

$\therefore \quad n-m = 1560-15 = 1545$

5. (1) : $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$=\lim\limits_{h\to 0}\dfrac{\{f(x)+f(h)-2xh-1\}-f(x)}{h}$ (Using the given relation)

$=\lim\limits_{h\to 0}(-2x)+\lim\limits_{h\to 0}\left(\dfrac{f(h)-1}{h}\right)$

$=\lim\limits_{h\to 0}(-2x)+\lim\limits_{h\to 0}\left(\dfrac{f(h)-f(0)}{h}\right)$

[Putting $x = 0 = y$ in the given relation we find $f(0) = f(0) + f(0) + 0 - 1 \Rightarrow f(0) = 1$]

$\therefore \quad f'(x)=-2x+f'(0)$

$\Rightarrow \quad f(x) = -x^2 - \sin\alpha\cdot x + C,$

As, $f(0) = -0-0+C \quad \Rightarrow C = 1$

$\therefore \quad f(x) = -x^2 - \sin\alpha\cdot x + 1$

So, $f\{f'(0)\}=f(-\sin\alpha)=-\sin^2\alpha+\sin^2\alpha+1$

$\therefore \quad f\{f'(0)\}=1$

6. (3) : $N=\displaystyle\sum_{r=1}^{10}(r-\omega)(r-\omega^2)=\sum_{r=1}^{10}(r^2+r+1)$

$=\dfrac{10\cdot 11\cdot 21}{6}+\dfrac{10\cdot 11}{2}+10=450=2\cdot 3^2\cdot 5^2$

The prime factors of N are 2, 3, 5.

7. (a,b,c) : At any point 't' on the given curve, we have

$\dfrac{dy}{dx}=\dfrac{3a\cos^2 t(-\sin t)}{3a\sin^2 t(\cos t)}=-\cot t$

The equations of the tangent and normal at 't' are

$x\cos t + y\sin t - (a/2)\sin 2t = 0$

and $x\sin t - y\cos t - a\cos 2t = 0$

$\therefore \quad OT=\dfrac{|(a/2)\sin 2t|}{\sqrt{\cos^2 t+\sin^2 t}}=(a/2)\sin 2t$

$\Rightarrow\ 2\cdot OT = a\sin 2t$

and $ON = \dfrac{|a\cos 2t|}{\sqrt{\sin^2 t+\cos^2 t}} = a\cos 2t$

Hence, $4OT^2 + ON^2 = a^2\sin^2 2t + a^2\cos^2 2t = a^2$

Length of the tangent at 't' $= \left|\dfrac{y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}{dy/dx}\right|$

$= \left|\dfrac{y\operatorname{cosec} t}{-\cot t}\right| = \left|\dfrac{y}{\cos t}\right|$

Length of the normal at 't' $= \left|y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\right|$

$= \left|y\sqrt{1+\cot^2 t}\right| = \left|\dfrac{y}{\sin t}\right|$

8. (a, b, c, d) : The point $\left(ct, \dfrac{c}{t}\right)$ lies on $x^2 + y^2 = a^2$

$\therefore\ c^2t^2 + \dfrac{c^2}{t^2} = a^2$ or $c^2t^4 - a^2t^2 + c^2 = 1$

If t_1, t_2, t_3, t_4 are the roots, then

$t_1+t_2+t_3+t_4 = 0, t_1t_2t_3t_4 = \dfrac{c^2}{c^2} = 1,$ ...(i)

$t_1t_2t_3 + t_2t_3t_4 + t_3t_4t_1 + t_4t_1t_2 = 0$ ...(ii)

(i), (ii) $\Rightarrow \dfrac{1}{t_1}+\dfrac{1}{t_2}+\dfrac{1}{t_3}+\dfrac{1}{t_4} = 0$ ...(iii)

$\therefore\ x_1 + x_2 + x_3 + x_4 = c(t_1 + t_2 + t_3 + t_4) = 0$

$y_1+y_2+y_3+y_4 = c\left(\dfrac{1}{t_1}+\dfrac{1}{t_2}+\dfrac{1}{t_3}+\dfrac{1}{t_4}\right) = 0$

$x_1\, x_2\, x_3\, x_4 = c^4\,(t_1\, t_2\, t_3\, t_4) = c^4$

and $y_1\, y_2\, y_3\, y_4 = \dfrac{c^4}{t_1t_2t_3t_4} = c^4$

9. (b, c) : The number of ways of 3 letters going to correct destinations is $\binom{7}{3} = 35$

The number of ways of 4 letters going to wrong destinations is

$4!\left(\dfrac{1}{2!}-\dfrac{1}{3!}+\dfrac{1}{4!}\right) = 9$

$\therefore$ The number of desired number is $35 \times 9 = 315$

10. (b, d) : Plane P_1 is parallel to $\vec{a}$ and $\vec{b}$. The normal to P_1 is along $\vec{a}\times\vec{b}$. Plane P_2 is parallel to $\vec{c}$ and $\vec{d}$. The normal to P_2 is along $\vec{c}\times\vec{d}$.

$\vec{A}$ is along the line of intersection of planes P_1 and P_2

$\therefore\ \vec{A}$ is along $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$

$\vec{a}\times\vec{b} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 2 & 3\\ 0 & 4 & -3\end{vmatrix} = -18\hat{i}$ and

$\vec{c}\times\vec{d} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 1 & -1\\ 3 & 3 & 0\end{vmatrix} = 3\hat{i} - 3\hat{j} - 3\hat{k}$

$\therefore\ \vec{A}$ is along $(-18\hat{i})\times(3\hat{i}-3\hat{j}-3\hat{k})$ *i.e.*, $54\hat{j} - 54\hat{k}$

Let the angle between $\vec{A}$ and $2\hat{i} + \hat{j} - 2\hat{k}$ is θ

$\therefore\ \cos\theta = \pm\left(\dfrac{(54\hat{i}-54\hat{k})(2\hat{i}+\hat{j}-2\hat{k})}{\sqrt{(54)^2+(54)^2}\cdot\sqrt{(2)^2+(1)^2+(-2)^2}}\right)$

$= \pm\left(\dfrac{54+108}{3\cdot\sqrt{2}\cdot 54}\right) = \pm\dfrac{1}{\sqrt{2}} \Rightarrow \theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}.$

11. (a, d): The tangent at $P(x,y)$ is $Y - y - (X - x)y_1 = 0$

It meets x-axis at $A\left(x-\dfrac{y}{y_1}, 0\right)$ and y-axis at $B(0, y - xy_1)$

$ar(\Delta OAB) = 2 \Rightarrow OA\cdot OB = 4$

$\Rightarrow\ \left(x-\dfrac{y}{p}\right)(y - xp) = 4$, where $p = \dfrac{dy}{dx}$

$\Rightarrow\ (y - xp)^2 = -4p$

$\Rightarrow\ y - xp = 2\sqrt{-p} \Rightarrow y = xp + 2\sqrt{-p}$

The general solution is $y = cx + 2\sqrt{-c}$ ...(i)

When $x = 1, y = 1 \Rightarrow c = -1, y = 2 - x\ \therefore\ y(2) = 0$

Differentiating (i) w.r.t c, $0 = x - \dfrac{1}{\sqrt{-c}}$ ...(ii)

Eliminating c from (i) and (ii), $y = -\dfrac{1}{x}+\dfrac{2}{x} = \dfrac{1}{x}, y(2) = \dfrac{1}{2}$

12. (a, b, c, d) : $l+m+n = 0,\ 4l = \dfrac{2mn}{m+n}$

$\Rightarrow\ 2\,lm - mn + 2\,ln = 0$

By eliminating n, we get

$2\left(\dfrac{l}{m}\right)^2 - \dfrac{l}{m} - 1 = 0\ \therefore\ \dfrac{l_1}{m_1} = 1, \dfrac{l_2}{m_2} = -\dfrac{1}{2}$

On solving, we get

$(l_1, m_1, n_1) = \left(\dfrac{1}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}, \dfrac{-2}{\sqrt{6}}\right)$

and $(l_2, m_2, n_2) = \left(\dfrac{1}{\sqrt{6}}, \dfrac{-2}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}\right)$

$\dfrac{l_1}{l_2}+\dfrac{m_1}{m_2}+\dfrac{n_1}{n_2} = 1-\dfrac{1}{2}-2 = \dfrac{-3}{2}$

$$l_1l_2+m_1m_2+n_1n_2=\frac{1}{6}-\frac{2}{6}-\frac{2}{6}=\frac{-1}{2}$$

$$l_1m_1n_1+l_2m_2n_2=\frac{-2}{6\sqrt{6}}-\frac{2}{6\sqrt{6}}=\frac{-\sqrt{6}}{9}$$

$$\therefore\ (l_1+l_2)(m_1+m_2)(n_1+n_2)=\frac{2}{\sqrt{6}}\times\frac{-1}{\sqrt{6}}\times\frac{-1}{\sqrt{6}}$$

$$=\frac{2}{6\sqrt{6}}=\frac{\sqrt{6}}{18}$$

13. (5) : We have,

$1+\ 2+3+...+(n-2)\le 1224\le 3+4+...+n$

$$\Rightarrow\ \frac{(n-2)(n-1)}{2}\le 1224\le\frac{(n-2)}{2}(n+3)$$

Thus $n^2-3n-2446\le 0$ and $n^2+n-2454\ge 0$

we get $49<n<51$ $\quad\therefore\ n=50$

Now, $\frac{n(n+1)}{2}-(2k+1)=1224\quad\therefore\ k=25$

Hence, $k-20=5$.

14. (6) : Let x, y and z be the probabilities of E_1, E_2 and E_3. Then

$x(1-y)(1-z)=\alpha$; $y(1-x)(1-z)=\beta$;

$z(1-x)(1-y)=\gamma$

Also, $(1-x)(1-y)(1-z)=p$

On dividing, we get, $\frac{x}{1-x}=\frac{\alpha}{p}$. $\therefore\ x=\frac{\alpha}{\alpha+p}$ etc.

$$\frac{P(E_1)}{P(E_3)}=\frac{\frac{\alpha}{\alpha+p}}{\frac{\gamma}{\gamma+p}}=\frac{1+\frac{p}{\gamma}}{1+\frac{p}{\alpha}}$$

Again, $(\alpha-2\beta)p=\alpha\beta$; $(\beta-3\gamma)p=2\beta\gamma$

We have $\alpha p=\beta(\alpha+2p)$ and $3\gamma p=\beta(p-2\gamma)$

From above, $\frac{\alpha}{3\gamma}=\frac{\alpha+2p}{p-2\gamma}$

$$\Rightarrow\ \frac{\alpha+2p}{\alpha}=\frac{p-2\gamma}{3\gamma}\quad\Rightarrow\ 1+\frac{2p}{\alpha}=\frac{p}{3\gamma}-\frac{2}{3}$$

$$\Rightarrow\ \frac{5}{3}=\frac{p}{3\gamma}-\frac{2p}{\alpha}\quad\Rightarrow\quad 5=\frac{p}{\gamma}-\frac{6p}{\alpha}$$

$$\Rightarrow\ 6\left(1+\frac{p}{\alpha}\right)=1+\frac{p}{\gamma}.\ \therefore\ \frac{1+\frac{p}{\gamma}}{1+\frac{p}{\alpha}}=6$$

15. (2.7) : $\frac{x^2}{9}+\frac{y^2}{1}=1\ \Rightarrow\ A\equiv(3,0),B\equiv(0,1)$

The line $AB:\frac{x}{3}+\frac{y}{1}=1$ meets the circle $x^2+y^2=9$ at M.

$$\therefore\ \ x^2+\left(1-\frac{x}{3}\right)^2=9\ \Rightarrow\ \frac{10}{9}x^2-\frac{2x}{3}-8=0$$

$$\Rightarrow\ \ 5x^2-3x-36=0\ \Rightarrow\ x=-\frac{12}{5},\ y=\frac{9}{5}$$

$$\therefore\ \ M\equiv\left(-\frac{12}{5},\frac{9}{5}\right)$$

$$\text{Area of }\Delta AMO=\frac{1}{2}\begin{vmatrix}3 & 0 & 1\\ -\frac{12}{5} & \frac{9}{5} & 1\\ 0 & 0 & 1\end{vmatrix}=\frac{27}{10}=2.7$$

16. (485) : We can do casework on number of ladies and men to be invited.

X, Y can satisfy the condition in 4 ways.

(i) X invites 3 ladies and Y invites 3 men.

(ii) X invites 2 ladies, 1 man and Y invites 1 lady 2 men.

(iii) X invites 1 lady, 2 men and Y invites 2 ladies, 1 man.

(iv) X invites 3 men and Y invites 3 ladies.

The number of ways

$={}^4C_3\cdot{}^4C_3+{}^4C_2\cdot{}^3C_1\cdot{}^3C_1\cdot{}^4C_2+{}^4C_1\cdot{}^3C_2\cdot{}^3C_2\cdot{}^4C_1+{}^3C_3\cdot{}^3C_3$

$=16+324+144+1=485$

17. (11) : The number of non-negative integral solution of

$x+y+z=10$ is

$${}^{10+3-1}C_{3-1}={}^{12}C_2=\frac{12\cdot 11}{2}=66$$

Let $z=2k$, $k\ge 0$

We have $x+y+2k=10\ \ \Rightarrow\ \ x+y=10-2k$

Now, solution $={}^{10-2k+2-1}C_{2-1}={}^{11-2k}C_1=11-2k$

Now, $\sum_{k=0}(11-2k)=11+9+7+5+3+1=36$

$\therefore$ The required probability $=\frac{36}{66}=\frac{6}{11}$

18. (2.8) : $f(x)=\frac{A(x)}{B(x)}$, $A(x)=6(ax^2+bx+1)$,

$B(x)=px^2+qx+1$

Now, $f(2)=3, f(-2)=4$

$\Rightarrow\ A(2)=3B(2), A(-2)=4B(-2)$ ...(i)

$f'(2)=0, f'(-2)=0$

$\Rightarrow\ A'(2)=3\,B'(2)$

and $A'(-2)=4B'(-2)$ ...(ii)

(i) and (ii) $\Rightarrow 2(4a+2b+1)=4p+2q+1$,

$3(4a-2b+1)=2(4p-2q+1)$,

$2(4a+b)=4p+q$,

$3(-4a+b)=2(-4p+q)$

Solving above equations, we get

$$a=\frac{1}{4},\ b=-3,\ p=\frac{1}{4},\ q=-5$$

$$\therefore\ \ f(x)=\frac{6(x^2-12x+4)}{(x^2-20x+4)}\ \text{ and }\ f(1)=\frac{14}{5}=2.8$$

OLYMPIAD CORNER

1. $a_1, \ldots, a_k, a_{k+1}, \ldots, a_n$ are positive numbers ($k < n$). Suppose that the values of $a_{k+1}, \ldots, a_n$ are fixed. How should one choose the values of $a_1, \ldots, a_n$ in order to minimize $\sum\limits_{i,j,i \neq j} \dfrac{a_i}{a_j}$?

2. Let m be a positive integer. Define the sequence $a_0, a_1, a_2, \ldots$ by $a_0 = 0$, $a_1 = m$ and $a_{n+1} = m^2 a_n - a_{n-1}$ for $n = 1, 2, 3, \ldots$. Prove that an ordered pair (a, b) of non-negative integers, with $a \leq b$, gives a solution to the equation, $\dfrac{a^2 + b^2}{ab+1} = m^2$ if and only if (a, b) is of the form (a_n, a_{n+1}) for some $n \geq 0$.

3. In a ΔABC, $\angle C = 2\angle B$. P is a point in the interior of ΔABC satisfying that $AP = AC$ and $PB = PC$. Show that AP trisects $\angle A$.

4. Determine all the possible values of the sum of the digits of the perfect squares.

5. $ABCD$ is a convex quadrilateral and O is the intersection of its diagonals. Let L, M, N be the mid-points of DB, BC, CA respectively. Suppose that AL, OM, DN are concurrent. Show that either $AD \parallel BC$ or $[ABCD] = 2[OBC]$.

SOLUTIONS

1. To minimize the given rational function, choose

$$a_i = \left(\frac{a_{k+1} + \ldots + a_n}{\dfrac{1}{a_{k+1}} + \ldots + \dfrac{1}{a_n}} \right)^{1/2} = (\mathrm{A} \cdot \mathrm{H})^{1/2}, i = 1, 2, \ldots, k$$

where A is the arithmetic mean and H is the harmonic mean of $a_{k+1}, \ldots, a_n$.

To prove this, we will be forgiven if we change notation : let $x_i = a_i$, $i = 1, 2, \ldots, k$ and $b_r = a_{k+r}$, $r = 1, \ldots, m$ with $k + m = n$ and denote the given rational function $F(x_1, \ldots, x_k)$.

Then we have $F(x_1, \ldots, x_k) = X + Y + B$, where

$$X = \sum_{1 \leq i < j \leq k} \left(\frac{x_i}{x_j} + \frac{x_j}{x_i} \right),$$

$$Y = \sum_{1 \leq i \leq k} \; \sum_{1 \leq r \leq m} \left(\frac{x_i}{b_r} + \frac{b_r}{x_i} \right),$$

$$B = \sum_{1 \leq r < s \leq m} \left(\frac{b_r}{b_s} + \frac{b_s}{b_r} \right).$$

Note that B is fixed and Y can be improved to

$$Y = \sum_{1 \leq i \leq k} \left(\left(\sum_{1 \leq r \leq m} \frac{1}{b_r} \right) x_i + \left(\sum_{1 \leq r \leq m} b_i \right) \frac{1}{x_i} \right)$$

$= \sum\limits_i \left(\dfrac{m}{\mathrm{H}} x_i + \dfrac{m\mathrm{A}}{x_i} \right)$, where A is the arithmetic mean and H is the harmonic mean of the b_r.

Now we recall that the simple function $\alpha x + \dfrac{\beta}{x}$ (with α, β, x all positive) assumes its minimum when $\alpha x = \dfrac{\beta}{x}$; that is $x = \sqrt{\beta / \alpha}$. Thus each of the terms in Y (and so Y itself) assumes its minimum when we choose, for $i = 1, 2, \ldots, k$, $x_i = \sqrt{\dfrac{m\mathrm{A}}{(m/\mathrm{H})}} = \sqrt{\mathrm{AH}}$, as asserted.

But there is more. It is also known that each term in X, (and so X itself) assumes its minimum when $x_i = x_j$, with $1 \leq i < j \leq k$. Thus choosing all $x_i = \sqrt{\mathrm{AH}}$ minimizes both X and Y and, since B is fixed, minimizes $F(x_1, \ldots, x_k)$ as claimed.

2. Let us first prove by induction that

$$\frac{a_n^2 + a_{n+1}^2}{a_n \cdot a_{n+1} + 1} = m^2 \text{ for all } n \ge 0.$$

Proof : Base case $(n = 0)$: $\frac{a_0^2 + a_1^2}{a_0 \cdot a_1 + 1} = \frac{0 + m^2}{0 + 1} = m^2$.

Now, let us assume that it is true for $n = k$, $k \ge 0$.

Then, $\frac{a_k^2 + a_{k+1}^2}{a_k \cdot a_{k+1} + 1} = m^2$

$\Rightarrow \quad a_k^2 + a_{k+1}^2 = m^2 \cdot a_k \cdot a_{k+1} + m^2$

$\Rightarrow \quad a_{k+1}^2 + m^4 a_{k+1}^2 - 2m^2 \cdot a_k \cdot a_{k+1} + a_k^2$
$= m^2 + m^4 a_{k+1}^2 - m^2 \cdot a_k \cdot a_{k+1}$

$\Rightarrow \quad a_{k+1}^2 + (m^2 a_{k+1} - a_k)^2$
$= m^2 + m^2 a_{k+1}(m^2 a_{k+1} - a_k)$

$\Rightarrow \quad a_{k+1}^2 + a_{k+2}^2 = m^2 + m^2 \cdot a_{k+1} \cdot a_{k+2}$

Therefore, $\frac{a_{k+1}^2 + a_{k+2}^2}{a_{k+1} \cdot a_{k+2} + 1} = m^2$, proving the induction. Hence (a_n, a_{n+1}) is a solution to $\frac{a^2 + b^2}{ab + 1} = m^2$ for all $n \ge 0$.

Now, consider the equation $\frac{a^2 + b^2}{ab + 1} = m^2$ and suppose $(a, b) = (x, y)$ is a solution with $0 \le x \le y$. Then

$$\frac{x^2 + y^2}{xy + 1} = m^2 \qquad ...(1)$$

If $x = 0$ then it is easily seen that $y = m$, so $(x, y) = (a_0, a_1)$. Since we are given $x \ge 0$, suppose now that $x > 0$.

Let us show that $y \le m^2 x$.

Proof by contradiction : Assume that $y > m^2 x$. Then $y = m^2 x + k$ where $k \ge 1$.

Substituting into (1) we get

$$\frac{x^2 + (m^2 x + k)^2}{(x)(m^2 x + k) + 1} = m^2$$

i.e., $x^2 + m^4 x^2 + 2m^2 xk + k^2 = m^4 x^2 + m^2 kx + m^2$

i.e., $(x^2 + k^2) + m^2(kx - 1) = 0$.

Now, $m^2(kx - 1) \ge 0$ since $kx \ge 1$ and $x^2 + k^2 \ge x^2 + 1 \ge 1$ so $(x^2 + k^2) + m^2(kx - 1) \ne 0$.

Thus we have a contradiction, so $y \le m^2 x$ if $x > 0$.

Now substitute $y = m^2 x - x_1$, where $0 \le x_1 < m^2 x$, into (1).

We have

$$\frac{x^2 + (m^2 x - x_1)^2}{x(m^2 x - x_1) + 1} = m^2$$

$x^2 + m^4 x^2 - 2m^2 x \cdot x_1 + x_1^2 = m^4 x^2 - m^2 x \cdot x_1 + m^2$

$x^2 + x_1^2 = m^2(x \cdot x_1 + 1)$

$$\frac{x^2 + x_1^2}{x \cdot x_1 + 1} = m^2 \qquad ....(2)$$

If $x_1 = 0$, then $x^2 = m^2$. Hence $x = m$ and $(x_1, x) = (0, m) = (a_0, a_1)$. But $y = m^2 x - x_1 = a_2$, so $(x, y) = (a_1, a_2)$.

Thus suppose $x_1 > 0$.

Let us now show that $x_1 < x$.

Proof by contradiction: Assume $x_1 \ge x$.

Then $m^2 x - y \ge x$, since $y = m^2 x - x_1$, and $\left(\frac{x^2 + y^2}{xy + 1}\right) x - y \ge x$, since (x, y) is a solution to $\frac{a^2 + b^2}{ab + 1} = m^2$.

So $x^3 + xy^2 \ge x^2 y + xy^2 + x + y$.

Hence $x^3 \ge x^2 y + x + y$, which is a contradiction since $y \ge x > 0$.

With the same proof that $y \le m^2 x$, we have $x \le m^2 x_1$. So the substitution $x = m^2 x_1 - x_2$ with $x_2 \ge 0$ is valid.

Substituting $x = m^2 x_1 - x_2$ into (2) gives $\frac{x_1^2 + x_2^2}{x_1 \cdot x_2 + 1} = m^2$.

If $x_2 \ne 0$, then we continue with the substitution $x_i = m^2_{x_{i+1}} - x_{i+2}$ (*) until we get $\frac{x_j^2 + x_{j+1}^2}{x_j \cdot x_{j+1} + 1} = m^2$ and $x_{j+1} = 0$. (The sequence x_i is decreasing, non-negative and integer.)

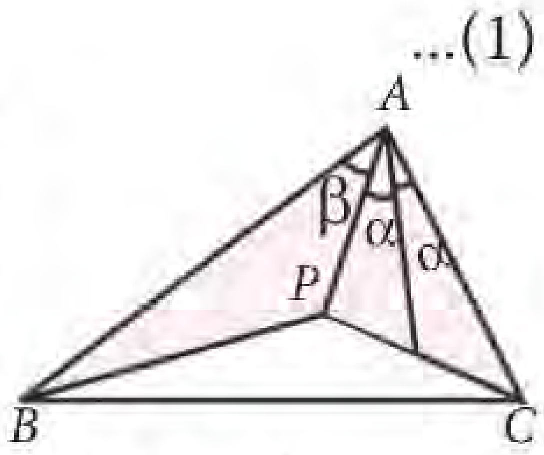
So, if $x_{j+1} = 0$, then $x_j^2 = m^2$ so $x_j = m$ and $(x_{j+1}, x_j) = (0, m) = (a_0, a_1)$.

Then $(x_j, x_{j-1}) = (a_1, a_2)$ since $x_{j-1} = m^2 x_j - x_{j+1}$ (from (*)).

Continuing, we have $(x_1, x) = (a_{n-1}, a_n)$ for some n. Then $(x, y) = (a_n, a_{n+1})$.

Hence $\frac{a^2 + b^2}{ab + 1} = m^2$ has solutions (a, b) if and only if $(a, b) = (a_n, a_{n+1})$ for some n.

3. Let $\angle PAC$ and $\angle BAP$ be 2α and β respectively. Then, since $\angle C = 2\angle B$, we deduce from $A + B + C = 180°$ that
$2\alpha + \beta + 3B = 180°$. ...(1)



The angles at the base of the isosceles triangle PAC are each $90° - \alpha$. Also ΔBPC is isosceles, having base angles $C - (90° - \alpha) = 2B + \alpha - 90°$, and so $\angle BPA = 180° - (\angle PBA + \angle BAP)$
$= 180° - [B - (2B + \alpha - 90°) + 180° - 2\alpha - 3B]$
$= 4B + 3\alpha - 90°$

As usual, let a, b and c denote the lengths of the sides BC, AC and AB. By the Law of Cosines, applied to ΔBPA, where $\overline{PA} = b$ and $\overline{PB} = \overline{PC} = 2b \sin\alpha$,
$c^2 = b^2 + (2b \sin\alpha)^2 - 2\cdot b\cdot 2b \sin\alpha\cdot\cos(4B + 3\alpha - 90°)$,
so that
$c^2 = b^2 [1 + 4 \sin^2\alpha - 4 \sin\alpha \sin(4B + 3\alpha)]$ ...(2)

We now use the fact that $\angle C = 2\angle B$ is equivalent to the condition $c^2 = b(b + a)$.

Since $a = 2\cdot \overline{PC} \cdot \cos(2B + \alpha - 90°)$
$= 4b \sin\alpha \sin(2B + \alpha)$, we have
$c^2 = b^2 [1 + 4 \sin\alpha \sin(2B + \alpha)]$ ...(3)

Therefore, from (2) and (3), we get
$b^2 [1 + 4 \sin^2\alpha - 4 \sin\alpha \sin(4B + 3\alpha)]$
$= b^2 [1 + 4 \sin\alpha \sin(2B + \alpha)]$,
which simplifies to
$\sin\alpha - \sin(4B + 3\alpha) = \sin(2B + \alpha)$.
Since $\sin\alpha - \sin(4B + 3\alpha) = -2\cos(2B + 2\alpha)\sin(2B + \alpha)$, this equation may be rewritten as $\sin(2B + \alpha).[1 + 2\cos(2B + 2\alpha)] = 0$

Since, from (1), $2B + \alpha < 180°$, we must have $1 + 2\cos(2B + 2\alpha) = 0$, giving $\cos(2B + 2\alpha) = -1/2$; that is,
$2B + 2\alpha = 120°$ ...(4)

Since, again from (1), $2B + 2\alpha < 180°$
Finally, we may eliminate B between (1) and (4) to obtain $\alpha = \beta$. The result follows.

4. The squares can only be 0, 1, 4 or 7 mod 9.
Thus the sum of the digits of a perfect square cannot be 2, 3, 5, 6 or 8 mod 9, since the number itself would then be 2, 3, 5, 6 or 8 mod 9.

We shall show that the sum of the digits of a perfect square can take every value of the form 0, 1, 4 or 7 mod 9.

$(10^m - 1)^2 = 10^{2m} - 2\cdot 10^m + 1$
$$= \underbrace{99\ldots9}_{m-1}8\underbrace{0\ldots0}_{m-1}1,\ m \ge 1.$$

The sum of the digits is $9m$, giving all the values greater than or equal to 9 congruent to 0 mod 9

$(10^m - 2)^2 = 10^{2m} - 4\cdot 10^m + 4$
$$= \underbrace{99\ldots9}_{m-1}6\underbrace{0\ldots0}_{m-1}4,\ m \ge 1.$$

The sum of the digits is $9m + 1$, which gives all values greater than or equal to 10 congruent to 1 mod 9.

$(10^m - 3)^2 = 10^{2m} - 6\cdot 10^m + 9$
$$= \underbrace{99\ldots9}_{m-1}4\underbrace{0\ \ldots\ 0}_{m-1}9,\ m \ge 1.$$

The sum of the digits is $9m + 4$, which takes every value greater than or equal to 13 which is congruent to 4 mod 9

$(10^m - 5)^2 = 10^{2m} - 10^{m+1} + 25$
$$= \underbrace{9\ \ldots\ 9}_{m-1}\underbrace{00\ldots0}_{m-1}25.$$

The sum of the digits is $9(m - 1) + 7 = 9m - 2$, from which we get every value greater than or equal to 7 congruent to 7 mod 9.

We have taken care of all the integers apart from 0, 1, 4, which are the sums of the digits of 0^2, 1^2 and 2^2 respectively.

5. Let O be the origin of a coordinate system where A, B, C, D are represented by $(a, 0)$, $(0, b)$, $(c, 0)$, $(0, d)$ with a, b positive and c, d negative. Thus L is the point

$\left(0, \frac{(b+d)}{2}\right)$, M is $\left(\frac{c}{2}, \frac{b}{2}\right)$, N is $\left(\frac{(a+c)}{2}, 0\right)$ and

$AL : (b + d)x + 2ay - a(b + d) = 0$
$OM : bx - cy = 0$
$DN : 2dx + (a + c)y - d(a + c) = 0$.

These lines are concurrent if and only if
$$\begin{vmatrix} b & -c & 0 \\ b+d & 2a & -a(b+d) \\ 2d & a+c & -d(a+c) \end{vmatrix} = 0.$$

This equation reduces (after some manipulation) to $(ab - cd)[(a - c)(b - d) + 2bc] = 0$.

Consequently, either

(a) $ab = cd$, in which case $AD \parallel BC$, or

(b) $\frac{1}{2}(a-c)(b-d)\sin\alpha = 2\left(-\frac{1}{2}bc\sin\alpha\right)$

(where $\alpha = \angle AOB$), in which case $[ABCD] = 2[OBC]$.

◈◈

CONCEPT MAP

COMPLEX NUMBERS

Class XI

Complex Number

A number of the form $z = a + ib$, where $a, b \in R$ and $i = \sqrt{-1}$, is called a complex number. Here $\text{Re}(z) = a$ and $\text{Im}(z) = b$.

Algebra of Complex Numbers

Let $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$, then

- $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$
- $z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$
- $z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$
- $\dfrac{z_1}{z_2} = \dfrac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i\dfrac{(a_2 b_1 - a_1 b_2)}{a_2^2 + b_2^2}$, where $z_2 \neq 0$
- $z_1 = z_2 \Rightarrow a_1 = a_2$ and $b_1 = b_2$

Geometry of Complex Numbers

If z is a variable point and z_1, z_2 are two fixed points in the argand plane, then

- $|z - z_1| = |z - z_2|$ represents perpendicular bisector of the line segment joining z_1 and z_2.
- $|z - z_1| + |z - z_2| = K$ (a fixed quantity > 0) ...(i)
  - If $K > |z_1 - z_2|$, then (i) represents an ellipse.
  - If $K = |z_1 - z_2|$, then (i) represents the line segment joining z_1 and z_2.
  - If $K < |z_1 - z_2|$, then (i) does not represent any curve in the argand plane.
  - If $K \neq |z_1 - z_2|$, then $|z - z_1| - |z - z_2| = K$ represent a hyperbola with foci at z_1 and z_2.
  - If $K = |z_1 - z_2|$, then $|z - z_1| - |z - z_2| = K$ represents a straight line joining z_1 and z_2 but excluding the line segment joining z_1 and z_2.
- Triangle ABC with vertices $A(z_1)$, $B(z_2)$ and $C(z_3)$ is equilateral if and only if $\begin{vmatrix} 1 & z_1 & z_2 \\ 1 & z_2 & z_3 \\ 1 & z_3 & z_1 \end{vmatrix} = 0$.
- The equation of circle whose centre is at point having affix z_0 and radius R is $|z - z_0| = R$.
- The equation of circle whose centre is $-a$ and radius $R = \sqrt{|a|^2 - b}$ is $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$.

Some Basic Terms

If $z = a + ib$, then

- Conjugate : $\bar{z} = a - ib$
- Modulus : $|z| = \sqrt{a^2 + b^2}$
- Argument : $\arg(z) = \tan^{-1}(b/a)$

Properties

- $\overline{(\bar{z})} = z$
- $z = \bar{z} \Leftrightarrow z$ is purely real
- $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary
- $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$
- $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
- $\overline{(z_1 / z_2)} = \bar{z}_1 / \bar{z}_2; z_2 \neq 0$
- $\overline{(z^n)} = (\bar{z})^n$
- $|z| = |\bar{z}| = |-z| = |-\bar{z}|$
- $z\bar{z} = |z|^2 = |\bar{z}|^2$
- $|z^n| = |z|^n$, where $n \in Q$
- $|z_1 + z_2 + \ldots + z_n| \leq |z_1| + |z_2| + \ldots + |z_n|$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2n\pi \ \forall \ n \in I$
- $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2) + 2n\pi \ \forall \ n \in I$
- $\arg(z^n) = n \arg(z) + 2n\pi \ \forall \ n \in I$

Different Forms of Complex Numbers

- **Polar Form :** $z = a + ib = r(\cos\theta + i\sin\theta) = r \text{ cis } \theta$, where $r = \sqrt{a^2 + b^2}$, $\theta = \tan^{-1}(b/a)$
- **Euler's Form :** $z = re^{i\theta}$, $\bar{z} = re^{-i\theta}$, where $-\pi < \theta < \pi$, θ is the principal argument.

Square Root of a Complex Number

Let $z = a + ib$ be a complex number So, square root of $z = a + ib$ is defined as,

$$\sqrt{a + ib} = \pm\left\{\sqrt{\frac{1}{2}\left(\sqrt{a^2 + b^2} + a\right)} + i\sqrt{\frac{1}{2}\left(\sqrt{a^2 + b^2} - a\right)}\right\}$$

To find the square root of $a - ib$, replace i by $-i$ in the above result.

Cube Roots of Unity

Let $x = (1)^{1/3} \Rightarrow x^3 - 1 = 0 \Rightarrow x = 1, \dfrac{-1 + i\sqrt{3}}{2}, \dfrac{-1 - i\sqrt{3}}{2}$

or $x = 1, \omega, \omega^2$

Note : (i) $\omega^3 = 1$ (ii) $1 + \omega + \omega^2 = 0$

(iii) $1 + \omega^n + \omega^{2n} = \begin{cases} 0, & \text{if } n \text{ is not a multiple of } 3 \\ 3, & \text{if } n \text{ is a multiple of } 3 \end{cases}$

CONCEPT MAP

DETERMINANTS

Class XII

DETERMINANT

Corresponding to every square matrix A, there exists a number called the determinant of A and denoted by $|A|$.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $|A| = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$.

Properties of Determinants

- The value of a determinant remains unaltered if its rows and columns are interchanged.
- If two rows (or columns) of a determinant are inter-changed, the value of the determinant is multiplied by -1.
- If any two rows (or columns) of a determinant are identical, then the value of determinant is zero.
- If the elements of a row (or column) of a determinant are multiplied by any scalar, then the value of the new determinant is equal to same scalar times the value of the original determinant.
- If each element of any row (or column) of a determinant is the sum of two numbers, then the determinant is expressible as the sum of two determinants of the same order.

Minors and Cofactors

- For any matrix $A = [a_{ij}]_{n \times n}$, if we leave the row and the column of the element a_{ij}, then the value of determinant thus obtained is called the minor of a_{ij} and it is denoted by M_{ij}.
- The minor M_{ij} multiplied by $(-1)^{i+j}$ is called the cofactor of the element a_{ij} and it is denoted by A_{ij}.
  $\therefore A_{ij} = (-1)^{i+j} M_{ij}$

Adjoint of a Matrix

Adjoint of a matrix A, denoted by adj A, is defined as the transpose of the cofactors matrix of A.

Properties : If A is non-singular matrix of order n, then

- $A(\text{adj } A) = (\text{adj } A)A = |A| I_n$
- $|A \text{ (adj) } A| = |A|^n$
- $\text{adj}(AB) = (\text{adj } B) \cdot (\text{adj } A)$
- $|\text{adj } A| = |A|^{n-1}$
- $\text{adj}(\text{adj } A) = |A|^{n-2} A$
- $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$

Area of a Triangle

- Let ABC be a triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then area of ΔABC is
  $\Delta = \dfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$
- The area of the triangle formed by three collinear points is zero.

Singular and Non-Singular Matrices

A square matrix A of order n is said to be

- **Singular** if $|A| = 0$
- **Non-singular** if $|A| \neq 0$
- If A and B are non-singular matrices of the same order, then AB and BA are also non singular matrices of the same order.

Inverse of a Matrix

For any square matrix A, inverse of A is defined as $A^{-1} = \dfrac{1}{|A|}(\text{adj } A), |A| \neq 0$

Properties :

- $(A^{-1})^{-1} = A$
- $(A')^{-1} = (A^{-1})'$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

Solution of System of Linear Equations

Let $AX = B$ be the given system of equations:

- If $|A| \neq 0$, the system is consistent and has unique solution.
- If $|A| = 0$ and $(\text{adj } A)B \neq O$, then the system is inconsistent and hence it has no solution.
- If $|A| = 0$ and $(\text{adj } A)B = O$, then the system may be either consistent or inconsistent according as the system has either infinitely many solutions or no solution.

Exam Dates 2021
24^{th} to 28^{th} May

Challenging PROBLEMS ON JEE Main

1. For a positive integer n, a local extremum value of the function $f(x)=\left(1+x+\frac{x^2}{2!}+.......+\frac{x^n}{n!}\right)e^{-x}$ is

(a) 0 (b) 1 (c) $\frac{1}{e}$ (d) e

2. For $x, y, z \in R$, we have $[x]-y=2[y]-z=3[z]-x=\frac{5}{21}$, where $[\cdot]$ denotes G.I.F., then $x+y+z=$

(a) $\frac{41}{7}$ (b) $\frac{44}{7}$ (c) $-\frac{41}{7}$ (d) $-\frac{44}{7}$

3. If $f(x)$ is an increasing function from $R \to R$ such that $f''(x)>0$, $f(x)\neq 0$ and f^{-1} exists, then $\frac{d^2 f^{-1}(x)}{dx^2}$ is

(a) 0 (b) <0
(c) >0 (d) depends on $f(x)$

4. In ΔABC, if $3\sin A+4\cos B=6$ and $4\sin B=1-3\cos A$, then $\angle C=$

(a) 30° (b) 60° (c) 120° (d) 150°

5. Let $a, b \in (0, \pi/2)$ and $\frac{\sin^4 a}{\sin^2 b}+\frac{\cos^4 a}{\cos^2 b}=1$, then

(a) $a=b$ (b) $a=2b$ (c) $b=2a$ (d) $a=3b$

6. The curve represented by the equation $x^2+y\cos^2\alpha=\frac{1}{2}x\sin 2\alpha$, $x\cos 2\alpha=-y\sin 2\alpha$, where α is a parameter, is

(a) a parabola (b) a circle
(c) an ellipse (d) a hyperbola

7. Given, area of $\Delta ABC=\frac{1}{2}$, then minimum value of $a^2+\text{cosec}A$ is

(a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) $\sqrt{4}$ (d) $\sqrt{5}$

8. If $z_1, z_2, z_3 \in C$ and $|z_1|=|z_2|=|z_3|=1$, $z_1+z_2+z_3=1$ and $z_1z_2z_3=1$ such that Im $(z_1)<\text{Im}(z_2)<\text{Im}(z_3)$, then $|z_1+z_2^2+z_3^3|=$

(a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) $\sqrt{4}$ (d) $\sqrt{5}$

9. a, b are complex numbers on a unit circle centered at origin with $a+b\neq 0$, then $\text{Im}\left(\frac{4ab}{(a+b)^2}\right)=$

(a) 0 (b) 1 (c) -1 (d) 2

10. The value of $L=\lim_{x\to 0^+}(x^{x^x}-x^x)$ is

(a) -1 (b) 0 (c) 1 (d) 2

11. If $f(x)=\lim_{m\to\infty}\frac{\sin(\pi x^2)+(x+2)^m\cdot\tan x}{x^2+(x+2)^m}$, then $\lim_{x\to -1}f(x)$ is

(a) 0 (b) $-\tan 1$
(c) $\tan 1$ (d) does not exist

12. The first term of a finite G.P. of real numbers is positive and the sum of the series is negative, then a possible number of terms in the series is

(a) 19 (b) 20 (c) 21 (d) 23

13. Let $H_n=1+\frac{1}{2}+\frac{1}{3}+.......+\frac{1}{n}$ and $T_n=\frac{1}{(n+1)H_n H_{n+1}}$, then $T_1+T_2+T_3+.....\infty=$

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{4}$

By : Tapas Kr. Yogi (Prime Axis Classes), Visakhapatnam Mob : 09533632105, 09949428844

14. If $I = \int \frac{x + \sin x - \cos x - 1}{x + e^x + \sin x} = f(x) - \log(g(x)) + C$, then $g(0) - f(0) =$

(a) 1 (b) -1 (c) e (d) 0

15. For $k > 0$, $\lim\limits_{n \to \infty} \frac{1^k + 3^k + \ldots\ldots + (2n-1)^k}{n^{k+1}} =$

(a) $\frac{2^k}{k+1}$ (b) $\frac{2^{k+1}}{k}$ (c) $\frac{2^{k+1}}{k+1}$ (d) $\frac{2^k}{k}$

16. Let $f: [0, 2] \to R$,

$f(x) = \sqrt{x^3 + 2 - 2\sqrt{x^3 + 1}} + \sqrt{x^3 + 10 - 6\sqrt{x^3 + 1}}$,

then $\int f(x)dx =$

(a) $2x + c$ (b) $x^2 - c$
(c) $x^3 + c$ (d) $x^3 + x^2 - x + 3c$

17. For $n \in N$, the product

$\left(4 - \frac{2}{1}\right)\left(4 - \frac{2}{2}\right)\left(4 - \frac{2}{3}\right)\ldots\ldots\left(4 - \frac{2}{n}\right)$

(a) is an integer for all n
(b) is a rational number for all n
(c) is an irrational number
(d) is an integer only for $n \le 16$

18. Let $T = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n}$

and $Y = {}^nC_1 - \frac{1}{2}\ {}^nC_2 + \frac{1}{3} \cdot {}^nC_3 \ldots\ldots\ldots$, then

(a) $2T = Y$ (b) $T = Y$
(c) $T = 2Y$ (d) $Y = 3T$

19. 25 persons are seated at a round table. All choices being equally likely, a team of 3 persons are chosen. The probability that atleast two of the three had been sitting next to each other is

(a) $\frac{11}{46}$ (b) $\frac{12}{46}$ (c) $\frac{13}{46}$ (d) $\frac{15}{46}$

20. A rectangular parallelopiped has sides of length a, b, c. The shortest distance of the edge of length 'a' from the diagonal (not meeting it) is

(a) $\frac{ab}{\sqrt{a^2 + b^2}}$ (b) $\frac{bc}{\sqrt{b^2 + c^2}}$

(c) $\frac{ac}{\sqrt{a^2 + c^2}}$ (d) $\frac{abc}{\sqrt{a^2 + b^2 + c^2}}$

21. 1000 unit cubes $(1 \times 1 \times 1)$ are glued together to form a $10 \times 10 \times 10$ cube. The maximum number of unit cubes that are visible from a single point in space is

(a) 912 (b) 1729 (c) 271 (d) 173

22. The graph of the function $y = x^3 + ax + b$ has exactly three common points with the co-ordinate axes and they are vertices of a right triangle $(a, b, \in R)$. Then value of a is

(a) $2\sqrt{3}$ (b) $-2\sqrt{3}$ (c) $\frac{3}{\sqrt{2}}$ (d) $\frac{-3}{\sqrt{2}}$

23. The number of pairs (a, b) of real numbers such that $a + \log a = b$ and $b + \log b = a$ is/are

(a) 0 (b) 1 (c) 2 (d) 3

24. $\lim_{n\to\infty}\left(1-\frac{2}{2\cdot 3}\right)\left(1-\frac{2}{3\cdot 4}\right)......\left(1-\frac{2}{(n+1)(n+2)}\right)=$

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{4}$

25. Two tangents $x - 2y + 10 = 0$ and $x = 2$ of a parabola intersect the tangent at vertex at $(-2, 4)$ and $(2, 0)$ respectively, then length of latus rectum of the parabola is

(a) $\sqrt{2}$ (b) $2\sqrt{2}$ (c) $3\sqrt{2}$ (d) $4\sqrt{2}$

NUMERICAL VALUE TYPE

26. Let $x, y \in [-\pi/4, \pi/4]$ and $x^3 + \sin x = 2a$ and $4y^3 + \frac{1}{2}\sin 2y + a = 0$ then $\cos(x + 2y) =$ _______.

27. For positive x, the minimum value of $x^{1000} + x^{900} + x^{90} + x^6 + \frac{1996}{x}$ is _______ .

28. If circles $x^2 + y^2 = c$ (radius $= \sqrt{3}$) and $x^2 + y^2 + ax + by + c = 0$ (radius $= \sqrt{6}$) intersect at two points P and Q, then length of $(PQ)^2$ is _______.

29. If K is a positive integer such that $36 + K$, $300 + K$, $596 + K$ are the squares of three consecutive terms of an A.P., then number of such possible progressions is/are _______.

30. The number of values of 'n' such that $\log_n(2^{231})$ is an integer is ________ ($n \in N$ and $n \neq 1$).

SOLUTIONS

1. (b) : Here, $f'(x)=\frac{-x^n}{n!}e^{-x}$, $f'(x)=0 \to x=0$

If n is even, $f'(x) < 0$ for $x \neq 0$. So no extrema

If n is odd, $f'(x) > 0$ for $x < 0$ and $f'(x) < 0$ for $x > 0$

So, $f(x) = 1$ is a (global) maximum value of f.

2. (b) : Given $3[z] - x = \frac{5}{21} \Rightarrow x = 3[z] - \frac{5}{21}$

$\Rightarrow [x] = 3[z] - 1$

Similarly, $[y] = [x] - 1$, $[z] = 2[y] - 1$

Solving these equations, we get $[x] = 2$, $[y] = 1$, $[z] = 1$

So, $x = 3 - \frac{5}{21} = \frac{58}{21}, y = \frac{37}{21}, z = \frac{37}{21}$

$\therefore \quad x + y + z = \frac{44}{7}$

3. (b) : $f(x)$ is an increasing function $\Rightarrow f'(x) > 0$

Let $g(x) = f^{-1}(x)$, then $f(g(x)) = x$.

$\Rightarrow g'(x) = \frac{1}{f'(g(x))} \Rightarrow g'(x) > 0$

Differentiating again, we get

$$g''(x) = \frac{-1}{(f'(g(x)))^2} \times f''(g(x)) \times g'(x)$$

Hence, $g''(x) < 0$

4. (a) : Given, $3 \sin A + 4 \cos B = 6$ ...(i)

and $4 \sin B + 3 \cos A = 1$ ...(ii)

On squaring and adding, (i) and (ii) we get

$\sin(A+B) = \frac{1}{2}$ i.e., $\angle C = 30°$ or $150°$

But $\angle C = 150°$ is not possible since $\angle C = 150°$

$\Rightarrow \angle A < 30°$

$\Rightarrow 3\sin A + 4\cos B < \frac{3}{2} + 4 < 6$, which is a contradiction.

Hence, $\angle C = 30°$

5. (a) : From the given relation, there exists a $\theta \in (0, \pi/2)$ such that $\frac{\sin^2 a}{\sin b} = \sin\theta$ and $\frac{\cos^2 a}{\cos b} = \cos\theta$

On eliminating a we get, $\cos(b - \theta) = 1$ and $\cos 2a = \cos(b + \theta)$

$\Rightarrow b - \theta = 0$ and $b + \theta = 2a \Rightarrow \theta = a = b$.

6. (a) : Simplifying the given equation, we have

$$\sin 2\alpha = \frac{x(2x^2+y)}{x^2+y^2}, \cos 2\alpha = \frac{-y(2x^2+y)}{x^2+y^2}$$

Eliminating α, we get

$$\frac{x^2(2x^2+y)^2}{(x^2+y^2)^2} + \frac{y^2(2x^2+y)^2}{(x^2+y^2)^2} = 1$$

Simplifying, $4x^2 = 1 - 4y$, which is a parabola.

7. (d) : Given, area $= \frac{1}{2} = \frac{1}{2} bc \sin A \Rightarrow \operatorname{cosec} A = bc$

So, $a^2 + \operatorname{cosec} A = a^2 + bc = b^2 + c^2 - 2bc \cos A + bc$

$= b^2 + c^2 - 2bc\sqrt{1 - \sin^2 A} + bc$

$= b^2 + c^2 - 2\sqrt{b^2c^2 - 1} + bc$

$\geq 3bc - 2\sqrt{(bc)^2 - 1}$ [$\because$ A.M. $\geq$ G.M.]

Let $y = 3x - 2\sqrt{x^2 - 1} \Rightarrow 5x^2 - 6xy + y^2 + 4 = 0$

Discriminant $\geq 0 \Rightarrow y \geq \sqrt{5}$

8. (d) : $\Sigma z_1 z_2 = z_1 z_2 z_3 \left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right)$

$= \bar{z}_1 + \bar{z}_2 + \bar{z}_3 = 1$

So, z_1, z_2, z_3 are roots of the equation $t^3 - 1\cdot t^2 + 1\cdot t - 1 = 0$

i.e., $(t - 1)(t^2 + 1) = 0 \Rightarrow z_1 = -i, z_2 = 1, z_3 = i$

$\therefore \quad |-i + 1 + i^3| = |1 - 2i| = \sqrt{5}$

9. (a) : $|a| = |b| = 1$.

Let $\frac{a}{b} = t \Rightarrow |t| = 1 \Rightarrow \bar{t} = \frac{1}{t}$

Let $z = \frac{4ab}{(a+b)^2} \Rightarrow \frac{4}{z} = \frac{a}{b} + \frac{b}{a} + 2 = t + \bar{t} + 2$,

which is real

$\therefore \quad \text{Im}(z) = 0$

10. (a) : Let $x^x = y$, $\therefore L = \lim_{x\to 0^+} (x^y - y)$

Now, $\lim_{x\to 0^+} y = \lim_{x\to 0^+} x^x = e^{\lim_{x\to 0^+} x\log x} = e^{\lim_{x\to 0^+} \frac{\log x}{1/x}}$

$= e^{\lim_{x\to 0^+} \frac{1/x}{-1/x^2}} = e^{-0} = 1$

Hence, $L = (0 - 1) = -1$

11. (d) : Taking three cases, according to $x + 2 > 1$, $x + 2 < 1$ and $x + 2 = 1$, we have

$$f(x) = \begin{cases} \tan x, x+2>1 \text{ i.e., } x>1 \\ \frac{\sin(\pi x^2)}{x^2}, x+2<1 \cup x+2>0 \text{ i.e., } -2<x<-1 \end{cases}$$

Hence, R.H.L. $= \lim_{x\to -1^+} f(x) = \tan 1$

and L.H.L. $= \lim_{x\to -1^-} f(x) = 0$

i.e., R.H.L. $\neq$ L.H.L. $\therefore$ Limit does not exist

12. (b) : We have, $S = \frac{a(r^n - 1)}{r-1}$

If $r \geq 0$, then every term is non-negative and hence S is non-negative, which is a contradiction to given data. Hence, r must be negative $\Rightarrow r^n - 1$ must be +ve *i.e.*, n should be even.

13. (a) : Consider, $H_{n+1} = H_n + \frac{1}{n+1}$

Hence, $T_n = \frac{1}{H_n} - \frac{1}{H_{n+1}}$

So, $T_1 + T_2 + T_3 + \text{......} \infty = \frac{1}{H_1} = 1$

14. (a) : Adding and subtracting e^x in the numerator, we get

$$I = \int \frac{(x + e^x + \sin x) - (1 + \cos x + e^x)}{x + e^x + \sin x} dx$$

$$= \int \left(1 - \frac{1 + e^x + \cos x}{x + e^x + \sin x}\right) dx$$

$= x - \log(x + e^x + \sin x) + C$

$\therefore \quad f(x) = x,\ g(x) = x + e^x + \sin x$

So, $g(0) - f(0) = 1$

15. (a) : Rewrite the given limit as

$$\lim_{n\to\infty} \frac{2^k}{n} \cdot \left[\left(\frac{1}{2n}\right)^k + \left(\frac{3}{2n}\right)^k + \text{....} + \left(\frac{2n-1}{2n}\right)^k\right]$$

$$= \frac{2^k}{k+1}$$

16. (a) : Putting $x^3 + 1 = t$ in $f(x)$, we have

$f(x) = \sqrt{t + 1 - 2\sqrt{t}} + \sqrt{t + 9 - 6\sqrt{t}}$

$= |\sqrt{t} - 1| + |\sqrt{t} - 3|$

For, $x \in [0, 2] \Rightarrow \sqrt{t} \in [1, 3] \Rightarrow |\sqrt{t} - 1| = \sqrt{t} - 1$ and $|\sqrt{t} - 3| = 3 - \sqrt{t}$

Hence, $f(x) = 2$

$\Rightarrow \int f(x)dx = 2x + c$

17. (a) : $\prod_{K=1}^{n} \cdot \left(4 - \frac{2}{K}\right) = 2^n \frac{\prod(2K-1)}{n!}$

$= 2^n \cdot \frac{\prod(2K-1) \cdot \prod(2K)}{n!\ \prod(2K)}$

$= \frac{2^n \cdot (2n)!}{n! \cdot 2^n \cdot (n!)} = \frac{(2n)!}{(n!)^2} = {}^{2n}C_n$

= Integer for all positive n.

18. (b) : Notice that

$\frac{1 - (1-x)^n}{x} = {}^nC_1 - {}^nC_2 x + {}^nC_3 x^2 \text{........}$

Integrating from 0 to 1,

$\int_0^1 \frac{1 - y^n}{1 - y} dy = {}^nC_1 - {}^nC_2 \cdot \frac{1}{2} + {}^nC_3 \cdot \frac{1}{3} \text{.....}$

(By putting $y = 1 - x$)

$\Rightarrow \int_0^1 (1 + y + y^2 + \text{.....} y^{n-1}) dy$

$= {}^nC_1 - {}^nC_2 \cdot \frac{1}{2} + {}^nC_3 \cdot \frac{1}{3} \text{.......}$

or, $1 + \frac{1}{2} + \frac{1}{3} + \text{.......} = {}^nC_1 - \frac{{}^nC_2}{2} + \frac{{}^nC_3}{3} \text{......}$

i.e., $T = Y$

19. (a) : Total ways $= {}^{25}C_3$.

Favourable ways = (25×21) ways to choose exactly 2 adjacent persons + (25) ways to choose exactly 3 adjacent persons

So, required probability $= \dfrac{25 \times 21 + 25}{^{25}C_3} = \dfrac{11}{46}$

20. (b) : Let the edges be along axes and $O(0, 0, 0)$, $A(a, 0, 0)$, $B(0, b, 0)$, $D(a, 0, c)$ then

$$\overrightarrow{OA} = a\hat{i},\ \overrightarrow{BD} = a\hat{i} - b\hat{j} + c\hat{k}$$

So, shortest distance $= \dfrac{\overrightarrow{OB}\cdot(\overrightarrow{OA}\times\overrightarrow{BD})}{|\overrightarrow{OA}\times\overrightarrow{BD}|} = \dfrac{bc}{\sqrt{b^2+c^2}}$

21. (c) : When we look from a point so that we can see 3 faces, the number of visible unit cubes is maximum. In that case, the $9 \times 9 \times 9$ cube that is behind the 3 visible faces is hidden from view. Therefore, the number of visible cubes $= 10 \times 10 \times 10 - 9 \times 9 \times 9 = 271$

22. (d) : The condition in the question implies that $x^3 + ax + b = 0$ has a double root α and one simple root β, *i.e.*, $x^3 + ax + b = (x - \alpha)^2 (x - \beta)$.

Let $A(\alpha, 0)$, $B(\beta, 0)$, $C(0, b)$ (on y-axis) with $\angle ACB = 90°$, $\alpha\beta < 0$.

So, $OA \cdot OB = OC^2 \Rightarrow -\alpha\beta = b^2$

and $\alpha + \alpha + \beta = 0$, $b = -\alpha^2\beta$

$$\Rightarrow \quad \alpha = \frac{1}{\sqrt[4]{2}},\ \beta = -\sqrt[4]{8}$$

So, $a = \alpha^2 + 2\alpha\beta = \dfrac{-3}{\sqrt{2}}$

23. (b) : On simplifying, we get $\log a = -a + \dfrac{1}{a}$

Now, if $a > 1$ then L.H.S. > 0 R.H.S. < 0

and if $a < 1$ then L.H.S. < 0, R.H.S. > 0, *i.e.*, no solution in either case. So, only solution is $a = 1$.

i.e., $(a, b) = (1, 1)$

24. (c) : Using $1 - \dfrac{2}{K(K+1)} = \dfrac{(K+2)(K-1)}{K(K+1)}$

The given terms simplify to $\dfrac{1}{3}\cdot\dfrac{(n+3)}{(n+1)} \to \dfrac{1}{3}$ as $n \to \infty$.

25. (d) : Focus is point of intersection of lines $\perp r$ to lines $x - 2y + 10 = 0$, $x = 2$ and passing through $(-2, 4)$ and $(2, 0)$ respectively.

Hence, focus $= (0, 0) \Rightarrow$ Latus rectum $= 4\sqrt{2}$

26. (1) : Let $U = 2y$ then 2^{nd} equation also looks like 1^{st} equation. So let $f(x) = x^3 + \sin x$, then $f(x)$ is odd and strictly increasing in given interval.

So, $f(x) = f(-U) = 2a$

$\Rightarrow \quad x = -U = -2y \Rightarrow x + 2y = 0$

$\Rightarrow \cos(x + 2y) = 1$

27. (2000) : Using A.M. - G.M. inequality on $x^{1000}, x^{900}, x^{90}, x^6, \dfrac{1}{x}, \dfrac{1}{x}, \ldots$ (1996 times), we get

$$\frac{x^{1000} + x^{900} + x^{90} + x^6 + \dfrac{1996}{x}}{2000} \geq \left(x^{1996}\frac{1}{x^{1996}}\right)^{1/2000} = 1$$

28. (8) : Notice that the two circles are orthogonal.

Hence, length of $PQ = \dfrac{2r_1r_2}{\sqrt{r_1^2 + r_2^2}} = 2\sqrt{2} \Rightarrow PQ^2 = 8$

29. (2) : We have, $36 + K = (a - d)^2$, $300 + K = a^2$ and $596 + K = (a + d)^2$.

On solving, we get $d = \pm 4$

30. (8) : $\log_n(2^{231}) = 3\cdot 7\cdot 11\cdot\dfrac{\log 2}{\log n}$

For this to be an integer, possible values of n are $2, 2^3, 2^7, 2^{11}, 2^{3\times 7}, 2^{3\times 11}, 2^{7\times 11}$ and $2^{3\times 7\times 11}$.

PRACTICE PAPER

BITSAT

Exam date: 24^{th} to 30^{th} June 2021

1. Three numbers, the third of which being 12, form decreasing G.P. If the last term was 9 instead of 12, the three numbers would have formed an A.P. The common ratio of the G.P. is
(a) 1/3 (b) 2/3 (c) 3/4 (d) 4/5

2. In a plane there are 37 straight lines, of which 13 pass through the point A and 11 pass through the point B. Besides, no three lines pass through one point, no lines passes through both points A and B, and no two are parallel, then the number of intersection points the lines have is equal to
(a) 535 (b) 601 (c) 728 (d) 963

3. The value of the determinant
$$\begin{vmatrix} 1 & e^{i\pi/3} & e^{i\pi/4} \\ e^{-i\pi/3} & 1 & e^{2i\pi/3} \\ e^{-i\pi/4} & e^{-2i\pi/3} & 1 \end{vmatrix}, \text{ where } i=\sqrt{-1}, \text{ is}$$
(a) $2+\sqrt{2}$ (b) $-(2+\sqrt{2})$
(c) $-2+\sqrt{3}$ (d) $-2-\sqrt{3}$

4. A rifle man firing at a distant target and has only 10% chance of hitting it. The minimum number of rounds he must fire in order to have 50% chance of hitting it atleast once is
(a) 6 (b) 7 (c) 8 (d) 9

5. $\lim\limits_{x\to 0} \dfrac{1-\cos x \cos 2x \cos 3x}{\sin^2 2x}$ is equal to
(a) 7/2 (b) 7/3 (c) 7/4 (d) 7/5

6. If the tangent at the point P on the circle $x^2 + y^2 + 6x + 6y = 2$ meets the straight line $5x - 2y + 6 = 0$ at a point Q on the y-axis, then the length of PQ is
(a) 4 (b) $2\sqrt{5}$ (c) 5 (d) $3\sqrt{5}$

7. The points A, B and C represent the complex numbers z_1, z_2, $(1 - i)z_1 + iz_2$ (where $i = \sqrt{-1}$) respectively on the complex plane. The triangle ABC is
(a) isosceles but not right angled
(b) right angled but not isosceles
(c) isosceles and right angled
(d) none of these

8. The area of the figure bounded by the curves $y = |x - 1|$ and $y = 3 - |x|$ is
(a) 2 sq. units (b) 3 sq. units
(c) 4 sq. units (d) 1 sq. unit

9. The direction cosines of the line drawn from $P(-5, 3, 1)$ to $Q(1, 5, -2)$ is
(a) $(6, 2, -3)$ (b) $(2, -4, 1)$
(c) $(-4, 8, -1)$ (d) $\left(\frac{6}{7}, \frac{2}{7}, -\frac{3}{7}\right)$

10. $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$ is equal to
(a) 1/2 (b) $\cos \pi/8$ (c) 1/8 (d) $\frac{1+\sqrt{2}}{2\sqrt{2}}$

11. The two vectors $\{\vec{a}=2\hat{i}+\hat{j}+3\hat{k}, \vec{b}=4\hat{i}-\lambda\hat{j}+6\hat{k}\}$ are parallel if λ is equal to
(a) 2 (b) -3 (c) 3 (d) -2

12. Let $f(x) = \begin{cases} \dfrac{a|x^2-x-2|}{2+x-x^2}, & x<2 \\ b, & x=2 \\ \dfrac{x-[x]}{x-2}, & x>2 \end{cases}$
([·] denotes the greatest integer function)
If $f(x)$ is continuous at $x = 2$, then
(a) $a = 1, b = 2$ (b) $a = 1, b = 1$
(c) $a = 0, b = 1$ (d) $a = 2, b = 1$

13. Let $a, b, c \in R$ and $a \neq 0$. If α is a root of $a^2x^2 + bx + c = 0$, β is a root of $a^2x^2 - bx - c = 0$ and $0 < \alpha < \beta$, then the equation $a^2x^2 + 2bx + 2c = 0$ has a root γ that always satisfies
(a) $\gamma = \alpha$ (b) $\gamma = \beta$
(c) $\gamma = (\alpha + \beta)/2$ (d) $\alpha < \gamma < \beta$

14. The term independent of x in the expansion of $\left[\sqrt{\left(\frac{x}{3}\right)} + \sqrt{\left(\frac{3}{2x^2}\right)}\right]^{10}$ is
(a) 5/12 (b) 1
(c) 1/3 (d) None of these

15. If $a < 0$, the function $f(x) = e^{ax} + e^{-ax}$ is a monotonically decreasing function for values of x given by
(a) $x > 0$ (b) $x < 0$
(c) $x > 1$ (d) $x < 1$

16. The number of solutions of the equation $\tan x + \sec x = 2\cos x$ lying in the interval $[0, 2\pi]$ is
(a) 0 (b) 1 (c) 2 (d) 3

17. Let R be a relation on the set of all lines in a plane defined by $(l_1, l_2) \in R$ such that $l_1 \parallel l_2$ then R is
(a) reflexive only (b) symmetric only
(c) transitive only (d) equivalence

18. Let $P(n) = 5^n - 2^n$, $P(n)$ is divisible by 3λ, where λ and n both are odd positive integers then the least value of n and λ will be
(a) 13 (b) 11 (c) 1 (d) 5

19. The rank of $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is equal to
(a) 4 (b) 3 (c) 5 (d) 1

20. A single letter is selected at random from the word 'PROBABILITY'. The probability that it is a vowel, is
(a) 2/11 (b) 3/11
(c) 4/11 (d) none of these

21. Let $A \equiv \{1, 2, 3, 4\}$, $B \equiv \{a, b, c\}$, then number of functions from $A \to B$, which are not onto is
(a) 8 (b) 24 (c) 45 (d) 6

22. The minimum value of the function defined by $f(x) = \text{maximum } \{x, x + 1, 2 - x\}$ is
(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) $\frac{3}{2}$

23. For parabola $x^2 + y^2 + 2xy - 6x - 2y + 3 = 0$, the focus is
(a) $(1, -1)$ (b) $(-1, 1)$
(c) $(3, 1)$ (d) none of these

24. If $A = \{x : x = 4n + 1, \forall\ 2 \leq n \leq 6\}$, then number of subsets of A are
(a) 2^2 (b) 2^3 (c) 2^5 (d) 2^6

25. The mean deviation and S.D. about actual mean of the series $a, a + d, a + 2d, ..., a + 2nd$ are respectively
(a) $\frac{n(n+1)d}{2n+1}, \sqrt{\frac{n(n-1)}{3}} \cdot d$
(b) $\frac{n(n-1)}{3}, \frac{n(n+1)}{2n} \cdot d$
(c) $\frac{n(n+1)d}{(2n+1)}, \sqrt{\frac{n(n+1)}{3}} \cdot d$
(d) $\frac{n(n-1)d}{2n-1}, \sqrt{\frac{n(n-1)}{3}} \cdot d$

26. If the arithmetic progression whose common difference is non zero, the sum of first $3n$ terms is equal to the sum of the next n terms. Then the ratio of the sum of the first $2n$ terms to the next $2n$ terms is
(a) 1 : 5 (b) 2 : 3
(c) 3 : 4 (d) none of these

27. $\lim_{n\to\infty} \frac{(n!)^{1/n}}{n}$ equals
(a) e (b) e^{-1}
(c) 1 (d) none of these

28. A man of height 2 m walks directly away from a lamp of height 5 m, on a level road at 3 m/s. The rate at which the length of his shadow is increasing is
(a) 1 m/s (b) 2 m/s (c) 3 m/s (d) 4 m/s

29. If the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide, then the value of b^2 is
(a) 3 (b) 16 (c) 9 (d) 12

30. If $\sin^{-1} x + \sin^{-1} y = \frac{2\pi}{3}$, then $\cos^{-1} x + \cos^{-1} y$ is equal to
(a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) π

31. If the sum of the binomial coefficients in the expansion of $\left(x+\frac{1}{x}\right)^n$ is 64, then the term independent of x is equal to

(a) 10 (b) 20 (c) 40 (d) 60

32. Solution of the differential equation

$\sin y\frac{dy}{dx} = \cos y(1 - x\cos y)$ is

(a) $\sec y = x - 1 - ce^x$ (b) $\sec y = x + 1 + ce^x$

(c) $\sec y = x + e^x + c$ (d) none of these

33. The function $f(x) = |x^2 - 3x + 2| + \cos|x|$ is not differentiable at x is equal to

(a) -1 (b) 0 (c) 1 (d) 2

34. From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on a shelf so that the dictionary is always in the middle. Then the number of such arrangements is

(a) at least 500 but less than 750

(b) at least 750 but less than 1000

(c) at least 1000 (d) less than 500

35. A normal to $y^2 = 4ax$ at t touches $x^2 - y^2 = a^2$, then $(t^2 + 1)^3$ is

(a) < 0 (b) > 0

(c) ≤ 0 (d) nothing can be said

36. If $f: X \to Y$ defined by $f(x) = \sqrt{3}\sin x + \cos x + 4$ is one-one and onto, then Y is

(a) $[1, 4]$ (b) $[2, 5]$ (c) $[1, 5]$ (d) $[2, 6]$

37. If $\vec{a},\vec{b},\vec{c}$ be unit vectors such that $\vec{a}\perp\vec{b},\vec{c}$ is inclined at the same angle to both $\vec{a}$ and $\vec{b}$ and $\vec{c} = p\vec{a} + q\vec{b} + r(\vec{a}\times\vec{b})$, then which of the following is true?

(a) $p = q$ (b) $|p| \leq 1$

(c) $|q| \leq 1$ (d) All of these

38. Area of the region bounded by the curves, $y = e^x$, $y = e^{-x}$ and the straight line $x = 1$ is given by

(a) $(e - e^{-1} + 2)$ sq. units

(b) $(e - e^{-1} - 2)$ sq. units

(c) $(e + e^{-1} - 2)$ sq. units

(d) none of these

39. If the normal at the end of latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ passes through $(0, -b)$, then $e^4 + e^2$ (where e is eccentricity) equals

(a) 1 (b) $\sqrt{2}$ (c) $\frac{\sqrt{5}-1}{2}$ (d) $\frac{\sqrt{5}+1}{2}$

40. A variable plane passes through the fixed point (a, b, c) and meets the axes at A, B, C. The locus of the point of intersection of the planes through A, B, C and parallel to the coordinate planes is

(a) $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$ (b) $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1$

(c) $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=-2$ (d) $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=-1$

41. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit coplanar vectors, then the scalar triple product

$[2\vec{a}-\vec{b}\ \ 2\vec{b}-\vec{c}\ \ 2\vec{c}-\vec{a}]$ is equal to

(a) 0 (b) 1 (c) $-\sqrt{3}$ (d) $\sqrt{3}$

42. The total number of numbers that can be formed by using all the digits 1, 2, 3, 4, 3, 2, 1, so that the odd digits always occupy the odd places, is

(a) 3 (b) 6 (c) 9 (d) 18

43. Number of real roots of the equation

$\sqrt{x}+\sqrt{x-\sqrt{(1-x)}} = 1$ is

(a) 0 (b) 1 (c) 2 (d) 3

44. $\int\frac{xe^x}{(1+x)^2}dx$ is equal to

(a) $\frac{e^x}{x+1}+c$ (b) $e^x(x + 1) + c$

(c) $-\frac{e^x}{(x+1)^2}+c$ (d) $\frac{e^x}{1+x^2}+c$

45. If $y = \sqrt{x+\sqrt{y+\sqrt{x+\sqrt{y+...\infty}}}}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{1}{2y-1}$ (b) $\frac{y^2-x}{2y^3-2xy-1}$

(c) $2y - 1$ (d) none of these

Monthly Test Drive CLASS XI ANSWER KEY

1. (b)	2. (b)	3. (b)	4. (c)	5. (a)
6. (c)	7. (a,c)	8. (a,b)	9. (b)	10. (a,b,c,d)
11. (a,c)	12. (a,d)	13. (b)	14. (d)	15. (d)
16. (c)	17. (7)	18. (4)	19. (0)	20. (0)

SOLUTIONS

1. (b) : $\because$ Numbers a, b, 12 are in G.P.

$\therefore \quad b^2 = 12a \qquad$ ...(i)

and a, b, 9 are in A.P.

$\therefore \quad 2b = a + 9 \quad$ or $\quad a = 2b - 9 \qquad$ ...(ii)

From (i) and (ii), we get

$b^2 = 12(2b - 9) \Rightarrow b^2 - 24b + 108 = 0$

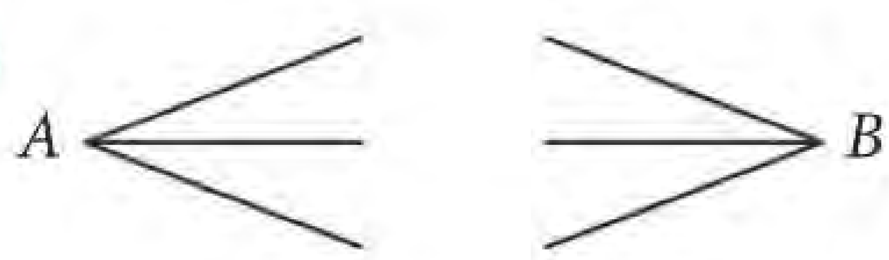
$\Rightarrow (b - 18)(b - 6) = 0 \quad \therefore \quad b = 6, 18$

From (ii), we get $a = 3, 27$

$\therefore$ Common ratio $= \frac{b}{a} = \frac{6}{3}$ and $\frac{18}{27} = 2$ and $\frac{2}{3}$

$\therefore$ Common ratio $= \frac{2}{3}$ (for decreasing G.P. common ratio $\neq 2$)

2. (a) :



13 pass through A 11 pass through B

$\therefore$ Number of intersection points

$= {}^{37}C_2 - {}^{13}C_2 - {}^{11}C_2 + 2 = 535$

($\because$ Two points A and B)

3. (b)

4. (b) : The probability of hitting in one shot

$= \frac{10}{100} = \frac{1}{10}$

If he fires n shots, the probability of hitting atleast once

$= 1 - \left(1 - \frac{1}{10}\right)^n = 1 - \left(\frac{9}{10}\right)^n = 50\% = \frac{50}{100} = \frac{1}{2}$

$\Rightarrow \left(\frac{9}{10}\right)^n = \left(\frac{1}{2}\right)$

$\therefore \quad n\{\log 9 - \log 10\} = \log 1 - \log 2$

$\Rightarrow \quad n\{2 \log 3 - 1\} = 0 - \log 2$

$\therefore \quad n = \frac{\log 2}{1 - 2\log 3} = \frac{0.3010300}{1 - 2 \times 0.4771213} = 6.6$

Hence, $n = 7$

5. (c)

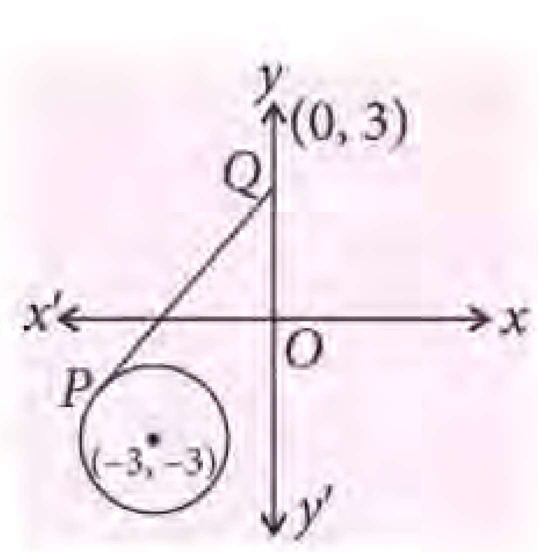
6. (c): $\because$ Q lies on y-axis,

Put $x = 0$ in

$5x - 2y + 6 = 0$

$\therefore \quad y = 3$

$\Rightarrow \quad Q \equiv (0, 3)$



$\therefore \quad PQ = \sqrt{0^2 + 3^2 + 0 + 6 \times 3 - 2} = \sqrt{25} = 5$

7. (c): Since, $A \equiv z_1, B \equiv z_2, C \equiv (1 - i)z_1 + iz_2$

$\therefore \quad AB = |z_1 - z_2|$,

$BC = |(1 - i)z_1 + iz_2 - z_2| = |1 - i|\,|z_1 - z_2|$

$= \sqrt{2}\,|z_1 - z_2|$

and $CA = |(1 - i)z_1 + iz_2 - z_1|$

$= |i|\,|-z_1 + z_2| = |z_1 - z_2|$

$\therefore \quad AB = CA$ and $(AB)^2 + (CA)^2 = (BC)^2$.

8. (c): Since, $y = |x - 1| = \begin{cases} x-1, & x \geq 1 \\ 1-x, & x < 1 \end{cases} \qquad$ ...(i)

and $y = 3 - |x| = \begin{cases} 3-x, & x \geq 0 \\ 3+x, & x < 0 \end{cases} \qquad$ ...(ii)

Solving (i) and (ii), we get

$x = 2$ and $x = -1$

$\therefore$ Required area $= \left|\int_{-1}^{2}(3 - |x| - |x - 1|)dx\right|$

$= \left|\int_{-1}^{0}(2x + 2)dx + \int_0^1 2dx + \int_1^2 (4 - 2x)dx\right|$

$= 4$ sq. units.

9. (d) : D.R.'s of PQ are $1 - (-5), 5 - 3, -2 - 1$

i.e., 6, 2, -3

and $\sqrt{6^2 + 2^2 + (-3)^2} = 7 \therefore$ D.C.'s are $\frac{6}{7}, \frac{2}{7}, \frac{3}{-7}$

10. (c): $(1 + \cos \pi/8)\,(1 + \cos 3\pi/8)\,(1 + \cos 5\pi/8)$ $(1 + \cos 7\pi/8)$

$= 2\cos^2(\pi/16) \cdot 2\cos^2(3\pi/16) \cdot 2\cos^2(5\pi/16) \cdot 2\cos^2(7\pi/16)$

$= 16[\cos(\pi/16)\cos(3\pi/16)\cos(5\pi/16)\cos(7\pi/16)]^2$

$= [2\cos(7\pi/16)\cos(\pi/16)]^2\,[2\cos(5\pi/16)\cos(3\pi/16)]^2$

$= [\cos(\pi/2) + \cos(3\pi/8)]^2\,[\cos(\pi/2) + \cos(\pi/8)]^2$

$= \cos^2(3\pi/8)\cos^2(\pi/8) = \frac{1}{4}(\cos \pi/2 + \cos \pi/4)^2 = \frac{1}{8}$

11. (d) : For parallel, $\vec{a} \times \vec{b} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 4 & -\lambda & 6 \end{vmatrix} = 0$$

$\Rightarrow \quad \hat{i}\,(6 + 3\lambda) - \hat{j}(0) + \hat{k}(-2\lambda - 4) = 0$

$\therefore \quad 6 + 3\lambda = 0 \Rightarrow \lambda = -2$

12. (b) : We have, $f(x) = \begin{cases} \dfrac{a|x^2 - x - 2|}{2 + x - x^2}, & x < 2 \\ b, & x = 2 \\ \dfrac{x - [x]}{x - 2}, & x > 2 \end{cases}$

$$f(x)=\begin{cases} -\dfrac{a(x^2-x-2)}{2+x-x^2}, & x<2 \\ b, & x=2 \\ \dfrac{x-[x]}{x-2}, & x>2 \end{cases}$$

$$\Rightarrow \ f(x)=\begin{cases} a, & x<2 \\ b, & x=2 \\ \dfrac{x-[x]}{x-2}, & x>2 \end{cases}$$

Now, L.H.L. = $\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} a = a$

R.H.L. = $\lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h)$

$= \lim_{h\to 0} \dfrac{2+h-[2+h]}{2+h-2} = \lim_{h\to 0} \dfrac{2+h-2}{2+h-2} = 1$

Value of the function $= f(2) = b$. Since $f(x)$ is continuous at $x = 2$

$\therefore$ L.H.L. = R.H.L. = Value of $f(x)$

$\therefore$ $a = b = 1$

13. (d) : Let $f(x) = a^2x^2 + 2bx + 2c$

From the question, $a^2\alpha^2 + b\alpha + c = 0$ and $a^2\beta^2 - b\beta - c = 0$

Now, $f(\alpha) = a^2\alpha^2 + 2b\alpha + 2c = b\alpha + c = -a^2\alpha^2$

$f(\beta) = a^2\beta^2 + 2b\beta + 2c = 3(b\beta + c) = 3a^2\beta^2$

But $0 < \alpha < \beta \Rightarrow \alpha, \beta$ are real

$\therefore$ $f(\alpha) < 0, f(\beta) > 0$

Hence, $\alpha < \gamma < \beta$.

14. (d) : In the given expansion, $(r+1)^{\text{th}}$ term $= T_{r+1}$

$= {}^{10}C_r \left(\sqrt{\dfrac{x}{3}}\right)^{10-r} \left(\sqrt{\dfrac{3}{2x^2}}\right)^r$

$= {}^{10}C_r \left(\dfrac{x}{3}\right)^{5-\frac{r}{2}} \left(\dfrac{3}{2x^2}\right)^{\frac{r}{2}} = {}^{10}C_r \dfrac{x^{5-\frac{r}{2}-r}}{3^{5-\frac{r}{2}-\frac{r}{2}} \cdot 2^{\frac{r}{2}}}$

For independent of x,

Put $5 - \dfrac{r}{2} - r = 0$

$\therefore$ $5 = \dfrac{3r}{2} \Rightarrow r = \dfrac{10}{3}$, impossible

$\because$ $r \neq$ whole number

15. (b) **16. (c)**

17. (d) : Let each line $\ell \in$ set of the lines (L)

(i) As $\ell \parallel \ell \Rightarrow (\ell, \ell) \in R \ \forall \ \ell \in L$

$\Rightarrow$ R is reflexive.

(ii) Let $\ell_1, \ell_2 \in L$ such that $(\ell_1, \ell_2) \in R$ then $\ell_1 \parallel \ell_2 \Rightarrow \ell_2 \parallel \ell_1 \Rightarrow (\ell_2, \ell_1) \in R$

$\therefore$ R is symmetric.

(iii) Let $\ell_1, \ell_2, \ell_3 \in L$ such that $(\ell_1, \ell_2) \in R$ and $(\ell_2, \ell_3) \in R$

$\Rightarrow$ $\ell_1 \parallel \ell_2 \parallel \ell_3 \Rightarrow (\ell_1, \ell_3) \in R \Rightarrow R$ is transitive.

Since a relation R, which is reflexive, symmetric and transitive is known as equivalence relation.

$\therefore$ Given relation is an equivalence relation.

18. (c): $P(n) = 5^n - 2^n$

Let $n = 1 \Rightarrow P(1) = 3\lambda = 3$

$\therefore$ $\lambda = 1$

Similarly $n = 5 \therefore P(5) = 5^5 - 2^5$

$= 3125 - 32 = 3093 = 3 \times 1031$

In this case, $\lambda = 1031$

Similarly, we can check the result for other cases and find that the least value of λ and n is 1.

19. (b) : Rank of diagonal matrix = Order of matrix = 3.

20. (b) : There are 11 letters in the word 'PROBABILITY' out of which 1 can be selected in ${}^{11}C_1$ ways.

$\therefore$ Exhaustive number of cases $= {}^{11}C_1 = 11$

There are three vowels viz AIO.

Therefore, favourable number of cases $= {}^3C_1 = 3$

Hence, the required probability $= \dfrac{3}{11}$

21. (c) : Total number of functions from $A \to B = 3^4 = 81$

Number of onto mappings

= Coefficient of x^4 in $4!(e^x - 1)^3$.

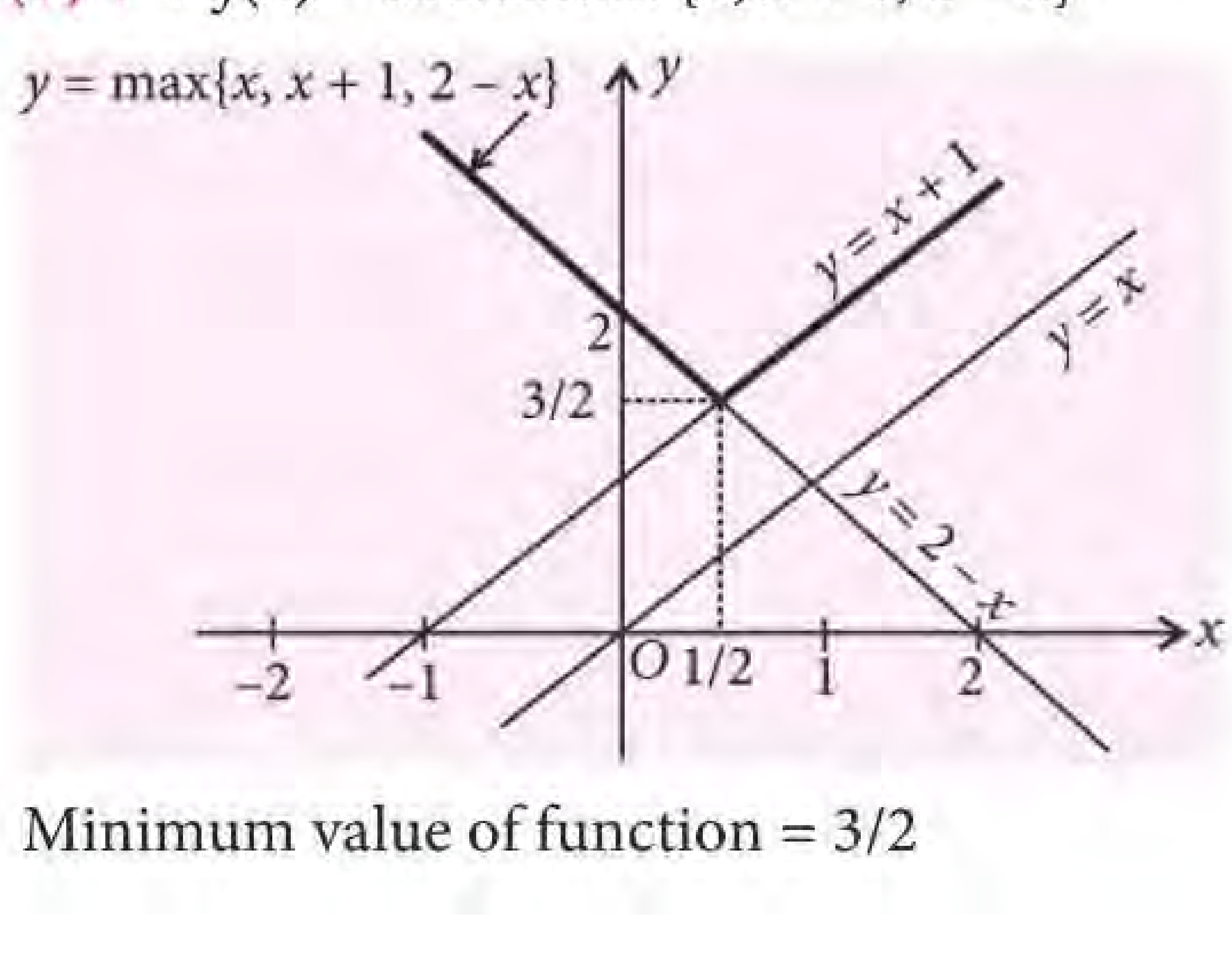
= Coefficient of x^4 in $4!(e^{3x} - 3e^{2x} + 3e^x - 1)$

$= 4!\left(\dfrac{3^4}{4!} - \dfrac{3\cdot 2^4}{4!} + \dfrac{3\cdot 1^4}{4!} - 0\right)$

$= 81 - 48 + 3 = 81 - 45 = 36$

$\therefore$ Number of functions from $A \to B$, which are not onto is $81 - 36 = 45$

22. (d) : $\because f(x) =$ maximum $\{x, x+1, 2-x\}$



$\therefore$ Minimum value of function = 3/2

23. (d)

24. (c): $A = \{x : x = 4n + 1 \ \forall\ 2 \le n \le 6\}$

or $\quad A = \{9, 13, 17, 21, 25\}$

$\Rightarrow$ Number of elements of $A = 5$

$\therefore$ Number of subsets of $A = 2^n = 2^5$

25. (c): Since, $\bar{x}$ = Mean of the series

$$= \frac{a+(a+d)+.....+(a+2nd)}{2n+1} = a + nd$$

x_i	$d = \mid x_i - \bar{x} \mid$	$\|d\|^2 = D$
a	nd	n^2d^2
$a + d$	$(n-1)d$	$(n-1)^2d^2$
$\vdots$	$\vdots$	$\vdots$
$a + (n-2)d$	$2d$	$4d^2$
$a + (n-1)d$	d	d^2
$a + nd$	0	0
$a + (n+1)d$	d	d^2
$a + (n+2)d$	$2d$	$4d^2$
$\vdots$	$\vdots$	$\vdots$
$a + 2nd$	nd	n^2d^2
	$\Sigma\|d\| = 2dn\left(\frac{n+1}{2}\right)$	$\Sigma\|d\|^2 = 2d^2 [1^2 + 2^2 + ... + n^2]$

We have $\Sigma|d| = n(n+1)\, d$ and

$$\Sigma|d^2| = \frac{2d^2\,(n)(n+1)(2n+1)}{6}$$

Now, M.D. $= \frac{\Sigma|d|}{N}$ and $\sigma^2 = \frac{\Sigma|d|^2}{N}$

M.D. $= \frac{n(n+1)d}{2n+1}$ and $\sigma^2 = \frac{2d^2\cdot n(n+1)(2n+1)}{6\cdot(2n+1)}$

$$= \frac{n(n+1)d^2}{3}$$

$\therefore$ S.D. $= \sqrt{\sigma^2} = \sqrt{\frac{n(n+1)}{3}}\cdot d$

26. (a) : Let $S_n = Pn^2 + Qn$ = Sum of first n terms

According to question,

Sum of first $3n$ terms = Sum of the next n terms

$\Rightarrow \quad S_{3n} = S_{4n} - S_{3n}$ or $2S_{3n} = S_{4n}$

or $\quad 2[P(3n)^2 + Q(3n)] = P(4n)^2 + Q(4n)$

$\Rightarrow \quad 2Pn^2 + 2Qn = 0$ or $Q = -nP$ $\quad$ ...(i)

Then, $\frac{\text{Sum of first } 2n \text{ terms}}{\text{Sum of next } 2n \text{ terms}} = \frac{S_{2n}}{S_{4n} - S_{2n}}$

$$= \frac{P(2n)^2 + Q(2n)}{[P(4n)^2 + Q(4n)] - [P(2n)^2 + Q(2n)]}$$

$$= \frac{2nP + Q}{6Pn + Q} = \frac{nP}{5nP} = \frac{1}{5} \qquad \text{[From (i)]}$$

27. (b) : Let $P = \lim_{n\to\infty} \frac{(n!)^{1/n}}{n} = \lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{1/n}$

$$= \lim_{n\to\infty}\left(\frac{1\cdot 2\cdot 3\cdot 4.....n}{n\cdot n\cdot n......n}\right)^{1/n}$$

$$= \lim_{n\to\infty}\left(\left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\left(\frac{3}{n}\right).....\left(\frac{n}{n}\right)\right)^{1/n}$$

$$\therefore \ln P = \lim_{n\to\infty}\frac{1}{n}\left[\ln\left(\frac{1}{n}\right) + \ln\left(\frac{2}{n}\right) + \ln\left(\frac{3}{n}\right) + ... + \ln\left(\frac{n}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ln\left(\frac{r}{n}\right) = \int_0^1 \ln x\, dx$$

$= [\ln x\cdot x - x]_0^1 = 0 - 1 = -1 \ \therefore\ P = e^{-1}$

28. (b)

29. (b) : If eccentricities of ellipse and hyperbola are e and e_1.

$\therefore$ Foci are $(\pm ae, 0)$ and $(\pm a_1e_1, 0)$.

Here, $ae = a_1e_1 \quad \Rightarrow \quad a^2e^2 = a_1^2e_1^2$

$$a^2\left(1 - \frac{b^2}{a^2}\right) = a_1^2\left(1 + \frac{b_1^2}{a_1^2}\right)$$

$\Rightarrow \quad a^2 - b^2 = a_1^2 + b_1^2 \ \Rightarrow 25 - b^2 = \frac{144}{25} + \frac{81}{25} = 9$

$\therefore \quad b^2 = 16$

30. (b) : $\cos^{-1}x + \cos^{-1}y = \frac{\pi}{2} - \sin^{-1}x + \frac{\pi}{2} - \sin^{-1}y$

$$= \pi - (\sin^{-1}x + \sin^{-1}y) = \pi - \frac{2\pi}{3} = \frac{\pi}{3} \qquad \text{(given)}$$

31. (b) : $\left(1 + \frac{1}{1}\right)^n = 64 \quad \Rightarrow \quad 2^n = 2^6 \quad \therefore \quad n = 6$

General term $T_{r+1} = {}^nC_r(x)^{n-r}\left(\frac{1}{x}\right)^r = {}^6C_r x^{6-2r}$

For independent of x,

$\quad 6 - 2r = 0 \quad \Rightarrow \quad r = 3$

$\therefore \quad T_{3+1} = {}^6C_3x^0 = {}^6C_3 = 20$

32. (b) : $\because\ \sin y\frac{dy}{dx} = \cos y(1 - x\cos y)$

$\Rightarrow \quad \tan y \dfrac{dy}{dx} = 1 - x\cos y$

$\Rightarrow \quad \sec y \tan y \dfrac{dy}{dx} - \sec y = -x$ ...(i)

Put $\sec y = v \quad \Rightarrow \quad \sec y \tan y \dfrac{dy}{dx} = \dfrac{dv}{dx}$

Then, from (i), $\dfrac{dv}{dx} - v = -x$

$\therefore \quad$ I.F. $= e^{\int -1 \cdot dx} = e^{-x}$

Then, the solution is

$v \cdot (e^{-x}) = \int (-x)e^{-x} dx$

$\Rightarrow \quad v \cdot e^{-x} = (-x)(-e^{-x}) + e^{-x} + c$

or $\quad v = x + 1 + ce^{x}$ or $\quad \sec y = x + 1 + ce^{x}$

33. (c)

34. (c): Out of 6 novels, 4 novels can be selected in ${}^6C_4$ ways.
Also out of 3 dictionaries, 1 dictionary can be selected in ${}^3C_1$ ways.
Since the dictionary is fixed in the middle, we only have to arrange 4 novels which can be done in 4! ways.

Then the number of ways $= {}^6C_4 \cdot {}^3C_1 \cdot 4! = \dfrac{6\cdot 5}{2} \cdot 3 \cdot 24 = 1080$

35. (a) : Normal at t, *i.e.* $(at^2, 2at)$ is
$tx + y = 2at + at^3$ or, $y = -tx + (2at + at^3)$
This will touch $x^2 - y^2 = a^2$

or $\quad \dfrac{x^2}{a^2} - \dfrac{y^2}{a^2} = 1$ if

$(2at + at^3)^2 = a^2(-t)^2 - a^2$ $\quad$ [Using $c^2 = a^2m^2 - b^2$]

$\Rightarrow \quad 4t^2 + t^6 + 4t^4 = t^2 - 1$

$\Rightarrow \quad t^6 + 3t^4 + 3t^2 + 1 = -t^4 \Rightarrow (t^2+1)^3 = -t^4 < 0$

[Here, if $t = 0$, then normal to $y^2 = 4ax$ will be x-axis at (0, 0) and x-axis can't touch rectangular hyperbola $x^2 - y^2 = a^2 \;\therefore\; t \neq 0$]

36. (d) : Rewrite $f(x) = 2\sin(x + \pi/6) + 4$

or $\quad f(x) = 2\cos\left(x - \dfrac{\pi}{3}\right) + 4$

$\therefore \quad Y = [2, 6]$ ($\because$ min and max values of $\sin\theta$ and $\cos\theta$ are -1 and $+1$)

37. (d) : $\because \quad |\vec{a}| = |\vec{b}| = |\vec{c}| = 1$

Let θ be the angle between $\vec{a}$ & $\vec{c}$ and $\vec{b}$ & $\vec{c}$

$\therefore \quad \cos\theta = \dfrac{\vec{a}\cdot\vec{c}}{|\vec{a}||\vec{c}|} = \vec{a}\cdot\vec{c}$ ...(i)

Similarly, $\cos\theta = \vec{b}\cdot\vec{c}$ ...(ii)

$\because \quad \vec{a} \perp \vec{b} \Rightarrow \vec{a}\cdot\vec{b} = 0$

$\because \quad \vec{c} = p\vec{a} + q\vec{b} + r(\vec{a}\times\vec{b})$

$\therefore \quad \vec{a}\cdot\vec{c} = p\vec{a}\cdot\vec{a} + q\,\vec{a}\cdot\vec{b} + r\,\vec{a}\cdot(\vec{a}\times\vec{b}) = p + 0 + 0 = p$

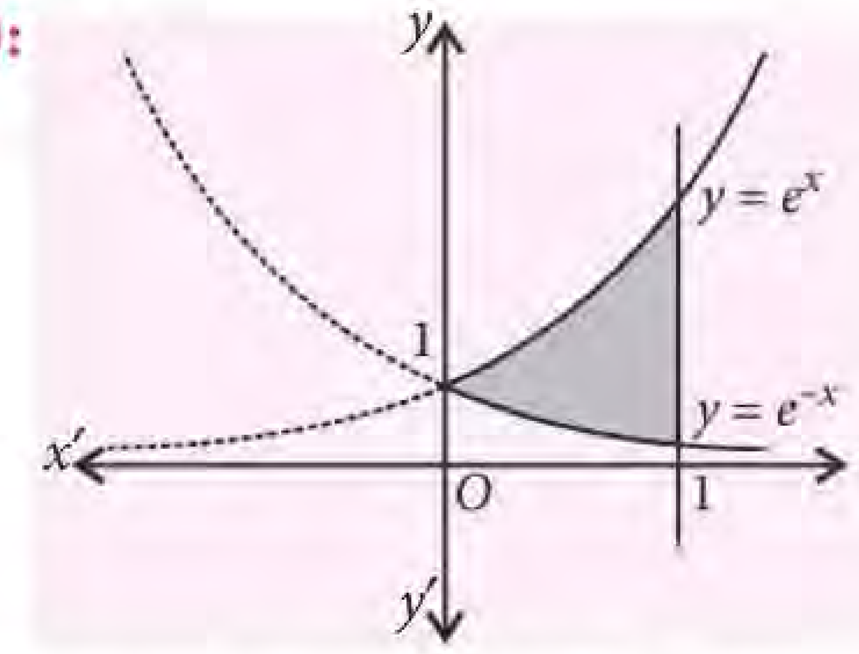
$\Rightarrow \quad \cos\theta = p \quad$ [from (i)]

$\therefore \quad |p| = |\cos\theta| \le 1$

Similarly, $\cos\theta = q \Rightarrow |q| \le 1$

Also, $p = q$

38. (c):



$\therefore \quad$ Required area $= \int_0^1 (e^x - e^{-x})dx$

$= [e^x + e^{-x}]_0^1 = (e + e^{-1}) - (e^0 + e^{-0})$

$= (e + e^{-1} - 2)$ sq. units.

PUZZLE CORNER

MATHDOKU

Introducing MATHDOKU, a mixture of ken-ken, sudoku and Mathematics. In this puzzle 6 × 6 grid is given, your objective is to fill the digits 1-6 so that each appear exactly once in each row and each column.
Notice that most boxes are part of a cluster. In the upper-left corner of each multibox cluster is a value that is combined using a specified operation on its numbers. For example, if that value is 3 for a two-box cluster and operation is multiply, you know that only 1 and 3 can go in there. But it is your job to determine which number goes where! A few cluster may have just one box and that is the number that fills that box.

12+		8+		5	10+
	3		4		
12+			9+		
10+		12+	3	1	9+
2	8+			7+	
		11+			2

Readers can send their responses at editor@mtg.in or post us with complete address. Winners' name with their valuable feedback will be published in next issue.

39. (a) : Normal at the extremity of latus rectum in the first quadrant $(ae, b^2/a)$ is

$$\frac{x-ae}{ae/a^2}=\frac{y-b^2/a}{b^2/ab^2}$$

As it passes through $(0, -b)$

$$\frac{-ae}{ae/a^2}=\frac{-b-b^2/a}{1/a}$$

$\Rightarrow\ -a^2 = -ab - b^2 \Rightarrow a^2 - b^2 = ab$

$\Rightarrow\ a^2e^2 = ab$ or $e^2 = b/a$

$\therefore\ e^4 = \dfrac{b^2}{a^2} = 1 = e^2 \Rightarrow e^4 + e^2 = 1$

40. (b) : Let plane is $\dfrac{x}{x_1}+\dfrac{y}{y_1}+\dfrac{z}{z_1}=1$

which passes through (a, b, c).

$\therefore\ \dfrac{a}{x_1}+\dfrac{b}{y_1}+\dfrac{c}{z_1}=1$

$\therefore$ Locus of (x_1, y_1, z_1) is $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}=1$

41. (a) : Given, $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

Also, $|\vec{a}| = 1$, $|\vec{b}| = 1$ and $|\vec{c}| = 1$

$\because\ [2\vec{a}-\vec{b}\ \ 2\vec{b}-\vec{c}\ \ 2\vec{c}-\vec{a}]$

$= (2\vec{a}-\vec{b})\cdot\{(2\vec{b}-\vec{c})\times(2\vec{c}-\vec{a})\}$

$= (2\vec{a}-\vec{b})\cdot\{4\vec{b}\times\vec{c}-2\vec{b}\times\vec{a}-2\vec{c}\times\vec{c}+\vec{c}\times\vec{a}\}$

$= (2\vec{a}-\vec{b})\cdot\{4(\vec{b}\times\vec{c})+2(\vec{a}\times\vec{b})-0+(\vec{c}\times\vec{a})\}$

$= 8\vec{a}\cdot(\vec{b}\times\vec{c})+4\vec{a}\cdot(\vec{a}\times\vec{b})+2\vec{a}\cdot(\vec{c}\times\vec{a})$

$\quad - 4\vec{b}\cdot(\vec{b}\times\vec{c})-2\vec{b}(\vec{a}\times\vec{b})-\vec{b}(\vec{c}\times\vec{a})$

$= 8[\vec{a}\ \vec{b}\ \vec{c}]+0+0-0-0-[\vec{a}\ \vec{b}\ \vec{c}]$

$= 7[\vec{a}\ \vec{b}\ \vec{c}] = 0 \qquad (\because [\vec{a}\ \vec{b}\ \vec{c}]=0)$

42. (d) : Required numbers $= \dfrac{4!}{2!2!}\times\dfrac{3!}{2!}=18$

43. (b) : Given $\sqrt{x}+\sqrt{x-\sqrt{(1-x)}}=1 \qquad$ ...(i)

$\Rightarrow\ \sqrt{x-\sqrt{(1-x)}}=1-\sqrt{x} \qquad$ ...(ii)

Squaring (ii) both sides, we get

$x - \sqrt{1-x} = 1 + x - 2\sqrt{x}$

$\Rightarrow\ -\sqrt{(1-x)} = 1 - 2\sqrt{x} \qquad$ ...(iii)

Squaring (iii) both sides, we get

$\Rightarrow\ 1 - x = 1 + 4x - 4\sqrt{x} \quad \Rightarrow \quad 4\sqrt{x} = 5x$

$\Rightarrow\ 25x^2 - 16x = 0 \quad \Rightarrow \quad x = 0, \dfrac{16}{25}$

$\Rightarrow\ x = \dfrac{16}{25} \qquad (\because x = 0$ does not satisfy (i))

44. (a) : Let $I = \displaystyle\int\frac{xe^x}{(1+x)^2}dx$

$$= \int e^x\left[\frac{(1+x)-1}{(1+x)^2}\right]dx = \int e^x\left\{\frac{1}{1+x}-\frac{1}{(1+x)^2}\right\}dx$$

$$= \frac{e^x}{(1+x)}+c \qquad \left(\because \int e^x\{f(x)+f'(x)\}dx = e^x f(x)+c\right)$$

45. (b) : We have, $y = \sqrt{x+\sqrt{y+y}}$

$\Rightarrow\ (y^2 - x) = \sqrt{2y} \quad \Rightarrow \quad (y^2 - x)^2 = 2y$

Differentiating both sides w.r.t. x, we get

$$2(y^2 - x)\left(2y\frac{dy}{dx}-1\right) = 2\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=\frac{(y^2-x)}{2y^3-2xy-1}$$

◆◆

SAMURAI SUDOKU

Samurai Sudoku puzzle consists of five overlapping sudoku grids. The standard sudoku rules apply to each 9 × 9 grid. Place digits from 1 to 9 in each empty cell. Every row, every column and every 3 × 3 box should contain one of each digit.

The puzzle has a unique answer.

3							7	8				9							3	5
1	7	4					5					4	5	7					9	
		8		1		4	6							3		1		8	7	
			1		5										6		3			
		5				6								8				7		
			4		3										2		4			
	1	6		9		2							1	6		3		5		
	2					3	9	1					7					3	6	9
4	9							6		1		3	9							8
									5		9									
								2				4								
									6		7									
2							3	4		9		6							1	8
7	4	4					2					7	3	1					9	
		9		7		1	6							8		5		2	3	
			1		6										8		1			
		2				5								9				6		
			9		7										2		7			
	4	7		9		8							8	2		3		9		
	2					6	4	7					9					8	7	5
1	8							3				5	6							3

Readers can send their responses at editor@mtg.in or post us with complete address. Winners' name with their valuable feedback will be published in next issue.

MONTHLY TEST DRIVE

This specially designed column enables students to self analyse their extent of understanding of all chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. If 1, $\log_9(3^{1-x}+2)$, $\log_3[4\cdot 3^x - 1]$ are in A.P. then x equals
   (a) $\log_3 4$ (b) $1-\log_3 4$
   (c) $1-\log_4 3$ (d) $\log_4 3$

2. Let A and B be two events such that
   $$P(\overline{A\cup B})=\frac{1}{6},\ P(A\cap B)=\frac{1}{4}\text{ and }P(\bar{A})=\frac{1}{4},$$
   where $\bar{A}$ stands for the complement of the event A. Then the events A and B are
   (a) equally likely but not independent
   (b) independent but not equally likely
   (c) independent and equally likely
   (d) none of these

3. Tangents drawn from the point $P(1, 8)$ to the circle $x^2+y^2-6x-4y-11=0$ touch the circle at the points A and B. The equation of the circumcircle of the triangle PAB is
   (a) $x^2+y^2+4x-6y+19=0$
   (b) $x^2+y^2-4x-10y+19=0$
   (c) $x^2+y^2-2x+6y-29=0$
   (d) $x^2+y^2-6x-4y+19=0$

4. The sum $\sum_{i=0}^{m}\binom{10}{i}\binom{20}{m-i}$ is maximum, when m is
   (a) 5 (b) 10 (c) 15 (d) 20

5. If $\log_{0.3}(x-1) < \log_{0.09}(x-1)$, then x lies in the interval
   (a) $(2,\infty)$ (b) $(1, 2)$
   (c) $(-2,-1)$ (d) none of these

6. The statement $p\to(q\to p)$ is equivalent to
   (a) $p\to(p\leftrightarrow q)$ (b) $p\to(p\to q)$
   (c) $p\to(p\vee q)$ (d) $p\to(p\wedge q)$

One or More Than One Option(s) Correct Type

7. Let $L=\lim_{x\to 0}\dfrac{a-\sqrt{a^2-x^2}-\dfrac{x^2}{4}}{x^4}$, $a>0$. If L is finite, then
   (a) $a=2$ (b) $a=1$
   (c) $L=\dfrac{1}{64}$ (d) $L=\dfrac{1}{32}$

8. An ellipse intersects the hyperbola $2x^2-2y^2=1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinate axes, then
   (a) Equation of ellipse is $x^2+2y^2=2$
   (b) The foci of ellipse are $(\pm 1, 0)$
   (c) Equation of ellipse is $x^2+2y^2=4$
   (d) The foci of ellipse are $(\pm\sqrt{2}, 0)$

PUZZLE CORNER

ANSWER - APRIL 2021

20× 5	2	5− 1	12+ 6	3	9+ 4
2	5 5	6	3	4	1
5− 1	11+ 4	5	2	2÷ 6	3
6	24× 3	2	4	4− 1	5
2− 4	6	1− 3	6+ 1	120× 5	2
3÷ 3	1	4	5	2	6

9. An n-digit number is a positive number with exactly n digits. Nine hundred distinct n-digit numbers are to be formed using only the three digits 2, 5 and 7. The smallest value of n for which this is possible is
(a) 6 (b) 7 (c) 8 (d) 9

10. If from a point $P(z_1)$ on the curve $|z| = 2$, pair of tangents are drawn on the curve $|z| = 1$ meeting $Q(z_2)$, $R(z_3)$, then
(a) complex number $\frac{z_1+z_2+z_3}{3}$ will lie on the curve $|z| = 1$
(b) $\left(\frac{4}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right)\left(\frac{4}{\bar{z}_1}+\frac{1}{\bar{z}_2}+\frac{1}{\bar{z}_3}\right)=9$
(c) orthocentre and circumcentre of ΔPQR will coincide
(d) $\arg\left(\frac{z_2}{z_3}\right)=\frac{2\pi}{3}$

11. All points lying inside the triangle formed by the points (1, 3), (5, 0) and (–1, 2) satisfy
(a) $3x + 2y \geq 0$ (b) $2x + y - 13 \geq 0$
(c) $2x - 3y - 12 \leq 0$ (d) $-2x + y \geq 0$

12. If $x^m \cdot y^n = (x + y)^{m+n}$, then dy/dx is
(a) $\frac{y}{x}$ (b) $\frac{x+y}{xy}$ (c) $(xy)^{-1}$ (d) $\left(\frac{x}{y}\right)^{-1}$

13. If in a triangle PQR, $\sin P$, $\sin Q$, $\sin R$ are in A.P. then
(a) the altitudes are in A.P.
(b) the altitudes are in H.P.
(c) the medians are in G.P.
(d) the medians are in A.P.

Comprehension Type

Let a, r, s, t be non-zero real numbers. Let $P(at^2, 2at)$, Q, $R(ar^2, 2ar)$ and $S(as^2, 2as)$ be distinct points on the parabola $y^2 = 4ax$. Suppose that PQ is the focal chord and lines QR and PK are parallel, where K is the point $(2a, 0)$.

14. The value of r is
(a) $-\frac{1}{t}$ (b) $\frac{t^2+1}{t}$ (c) $\frac{1}{t}$ (d) $\frac{t^2-1}{t}$

15. If $st = 1$, then the tangent at P and the normal at S to the parabola meet at a point whose ordinate is
(a) $\frac{(t^2+1)^2}{2t^3}$ (b) $\frac{a(t^2+2)^2}{t^3}$
(c) $\frac{a(t^2+1)^2}{t^2}$ (d) none of these

Matrix Match Type

16. Let $f(x) = \frac{x^2-6x+5}{x^2-5x+6}$.

	Column I		Column II
P.	If $-1 < x < 1$, then	1.	$f(x) > 0$
Q.	If $1 < x < 2$, then	2.	$f(x) < 0$
R.	If $x > 5$, then	3.	$0 < f(x) < 1$

	P	Q	R
(a)	2	1	3
(b)	3	1	2
(c)	3	2	1
(d)	2	3	1

Numerical Value Type

17. Let (x, y, z) be points with integer coordinates satisfying the system of homogeneous equations: $3x - y - z = 0; -3x + z = 0; -3x + 2y + z = 0$. Then the number of such points for which $x^2 + y^2 + z^2 \leq 100$ is ______.

18. Let ABC and ABC' be two non-congruent triangles with sides $AB = 4$, $AC = AC' = 2\sqrt{2}$ and angle $B = 30°$. The absolute value of the difference between the areas of these triangles is ______.

19. The largest value of the non-negative integer a for which $\lim_{x\to1}\left\{\frac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right\}^{\frac{1-x}{1-\sqrt{x}}}=\frac{1}{4}$ is ____.

20. Let $a_1, a_2, a_3, \ldots\ldots, a_{11}$ be real numbers satisfying $a_1 = 15$, $27 - 2a_2 > 0$ and $a_k = 2a_{k-1} - a_{k-2}$ for $k = 3, 4, \ldots, 11$. If $\frac{a_1^2+a_2^2+\ldots+a_{11}^2}{11}=90$, then the value of $\frac{a_1+a_2+\ldots+a_{11}}{11}$ is equal to _____.

Keys are published in this issue. Search now! ☺

SELF CHECK	Check your score! If your score is		
No. of questions attempted	> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
No. of questions correct	90-75%	GOOD WORK !	You can score good in the final exam.
Marks scored in percentage	74-60%	SATISFACTORY !	You need to score more next time.
	< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

CBSE warm-up!

CLASS-XII

Practice paper for CBSE Exam as per the reduced syllabus and marking scheme issued by CBSE for the academic session 2020-21.

Practice Paper 2021

Time Allowed : 3 hours
Maximum Marks : 80

GENERAL INSTRUCTIONS

(1) This question paper contains two parts A and B. Each part is compulsory. Part A carries 24 marks and Part B carries 56 marks.
(2) Part-A has Objective Type Questions and Part -B has Descriptive Type Questions.
(3) Both Part A and Part B have internal choices.

Part-A :

(1) It consists of two sections- I and II.
(2) Section I comprises of 16 very short answer type questions.
(3) Section II contains 2 case studies. Each case study comprises of 5 case-based MCQs. An examinee is to attempt any 4 out of 5 MCQs.

Part-B :

(1) It consists of three sections- III, IV and V.
(2) Section III comprises of 10 questions of 2 marks each.
(3) Section IV comprises of 7 questions of 3 marks each.
(4) Section V comprises of 3 questions of 5 marks each.
(5) Internal choice is provided in 3 questions of Section-III, 2 questions of Section-IV and 3 questions of Section-V. You have to attempt only one of the alternatives in all such questions.

PART-A

SECTION-I

All questions are compulsory. In case of internal choices attempt any one.

1. If $A = \{1, 2, 3\}$, $B = \{1, 4, 6, 9\}$ and R is a relation from A to B defined by 'x is greater than y', then find the range of R.

2. Find the adjoint of the matrix $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$.

OR

Find the value of $\begin{vmatrix}\cos 15^\circ & \sin 15^\circ\\ \sin 75^\circ & \cos 75^\circ\end{vmatrix}$.

3. A line makes angles α, β and γ with the co-ordinate axes. If $\alpha + \beta = 90^\circ$, then find the value of γ.

OR

Find the equation of the line passing through (x_1, y_1, z_1) having direction cosines l, m and n.

4. If $A = [a_{ij}]_{2\times 2}$, where $a_{ij} = \dfrac{(i+2j)^2}{2}$, then find A.

5. Write the value of $\int \dfrac{2-3\sin x}{\cos^2 x}\,dx$.

6. State the reason for the relation R in the set $\{1, 2, 3\}$ given by $R = \{(1, 2), (2, 1)\}$ not to be transitive.

7. Find the value of λ for which the vectors $2\hat{i}-\lambda\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}-\hat{k}$ are orthogonal.

OR

If $\vec{a} = 7\hat{i}+\hat{j}-4\hat{k}$ and $\vec{b} = 2\hat{i}+6\hat{j}+3\hat{k}$, then find the projection of $\vec{a}$ on $\vec{b}$.

8. Evaluate : $\int_2^3 3^x\,dx$

9. Write the direction cosines of the line whose equation is $5x - 3 = 15y + 7 = 3 - 10z$.

10. Write the integrating factor of the following differential equation :

$$(1+x^2)+(2xy-\cot x)\frac{dx}{dy}=0$$

OR

Write the sum of the order and degree of the following differential equation

$$\frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^3\right\}=0.$$

11. If matrix $A=\begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}$ is given to be symmetric, then find the values of a and b.

OR

If $A^T=\begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, then find $A^T - B^T$.

12. Find the value of 'p' for which the vectors $3\hat{i}+2\hat{j}+9\hat{k}$ and $\hat{i}-2p\hat{j}+3\hat{k}$ are parallel.

13. The coordinates of a point P are (3, 12, 4) w.r.t. origin O, then find the direction cosines of OP.

14. If O is origin and C is the mid point of $A(2, -1)$ and $B(-4, 3)$, then find the value of $\overrightarrow{OC}$.

15. Let R be an equivalence relation on a finite set A having n elements. Then, find the number of ordered pairs in R.

16. If a line makes angles θ_1, θ_2, θ_3 with the positive direction of co-ordinate axes respectively, then find the value of $\cos 2\theta_1 + \cos 2\theta_2 + \cos 2\theta_3$.

SECTION-II

Both the case study based questions are compulsory. Attempt any 4 sub-parts from each question 17 and 18. Each sub-part carries 1 mark.

17. Mr. Sahil is the owner of a high rise residential society having 50 apartments. When he set rent at ₹ 10000/month, all apartments are rented. If he increases rent by ₹ 250/month, one fewer apartment is rented. The maintenance cost for each occupied unit is ₹ 500/month.

Based on the above information answer the following questions.

(i) If P is the rent price per apartment and N is the number of rented apartment, then profit is given by
   (a) NP (b) $(N - 500)P$
   (c) $N(P - 500)$ (d) none of these

(ii) If x represent the number of apartments which are not rented, then the profit expressed as a function of x is
   (a) $(50 - x)(38 + x)$
   (b) $(50 + x)(38 - x)$
   (c) $250(50 - x)(38 + x)$
   (d) $250(50 + x)(38 - x)$

(iii) If $P = 10500$, then $N =$
   (a) 47 (b) 48
   (c) 49 (d) 50

(iv) If $P = 11,000$, then the profit is
   (a) ₹ 483000 (b) ₹ 500000
   (c) ₹ 505000 (d) ₹ 650000

(v) The rent that maximizes the total amount of profit is

(a) ₹ 11000 (b) ₹ 11500

(c) ₹ 15800 (d) ₹ 16500

18. Between students of class XII of two schools A and B basketball match is organised. For which, a team from each school is chosen, say T_1 be the team of school A and T_2 be the team of school B. These teams have to play two games against each other. It is assumed that the outcomes of the two games are independent. The probability of T_1 winning, drawing and losing a game against T_2 are $\frac{1}{2}, \frac{3}{10}$ and $\frac{1}{5}$ respectively.

Each team gets 2 points for a win, 1 point for a draw and 0 point for a loss in a game.

Let X and Y denote the total points scored by team A and B respectively, after two games.

Based on the above information, answer the following questions.

(i) $P(T_2$ winning a match against $T_1)$ is equal to

(a) 1/5 (b) 1/6

(c) 1/3 (d) none of these

(ii) $P(T_2$ drawing a match against $T_1)$ is equal to

(a) 1/2 (b) 1/3

(c) 1/6 (d) 3/10

(iii) $P(X > Y)$ is equal to

(a) 1/4 (b) 5/12

(c) 1/20 (d) 11/20

(iv) $P(X = Y)$ is equal to

(a) 11/100 (b) 1/3

(c) 29/100 (d) 1/2

(v) $P(X + Y = 8)$ is equal to

(a) 0 (b) 5/12

(c) 13/36 (d) 7/12

PART-B

SECTION-III

All questions are compulsory. In case of internal choices attempt any one.

19. Find the area enclosed by the line $y = 3x$, the x-axis, and the ordinates $x = 1$ and $x = 4$.

20. Find the value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - 3\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

OR

If $\sin[\cot^{-1}(x+1)] = \cos(\tan^{-1}x)$, then find x.

21. A random variable X has the following distribution.

X	1	2	3	4	5	6	7	8
$P(X)$	0.15	0.23	0.12	0.10	0.20	0.08	0.07	0.05

For the event $E = \{X$ is prime number$\}$ and $F = \{X < 4\}$, find $P(E \cup F)$.

22. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, then find A^{-1}.

23. Evaluate : $\int \frac{dx}{3\sin^2 x + 4}$

OR

Evaluate $\int_{\pi/3}^{\pi/4} (\tan x + \cot x)^2 dx$.

24. Determine the constants a and b such that the function

$$f(x) = \begin{cases} ax^2 + b, & \text{if } x > 2 \\ 2, & \text{if } x = 2 \\ 2ax - b, & \text{if } x < 2 \end{cases}$$

is continuous at $x = 2$.

25. Solve the differential equation $5\frac{dy}{dx} = e^x y^4$.

26. Find the equation of normal to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$ at $(1, 1)$.

27. A box contains N coins, of which m are fair and the rest are biased. The probability of getting head when a fair coin is tossed is $\frac{1}{2}$, while it is $\frac{2}{3}$ when a biased coin is tossed. A coin is drawn from the box at random and is tossed twice. The first time it shows head and the second time it shows tail. Find the probability that the coin drawn is fair.

28. If $\vec{a} = -3\hat{i} + n\hat{j} + 4\hat{k}$ and $\vec{b} = -2\hat{i} + 4\hat{j} + p\hat{k}$ are collinear, then find the value of n and p.

OR

If $\vec{a}=\hat{i}+\hat{j}+\hat{k}, \vec{b}=\hat{i}+\hat{j}, \vec{c}=\hat{i}$ and $(\vec{a}\times\vec{b})\times\vec{c}=\lambda\vec{a}+\mu\vec{b}$, then find the value of $\lambda + \mu$.

SECTION-IV

All questions are compulsory. In case of internal choices attempt any one.

29. Find the interval in which, the function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function.

30. If $\frac{dy}{dx} = \frac{2}{x+y}$, then show that $x + y + 2 = c \cdot e^{y/2}$.

31. Let $f(x)=\begin{cases} \sin x, & \text{for } x \geq 0 \\ 1-\cos x, & \text{for } x \leq 0 \end{cases}$ and $g(x) = e^x$.

Then, find the value of $(gof)'(0)$.

OR

Find the derivative of $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ with respect to $\cot^{-1}\left(\frac{1-3x^2}{3x-x^3}\right)$.

32. Find the area of the region $R = \{(x, y) : |x| \leq |y| \text{ and } x^2 + y^2 \leq 1\}$.

33. Let $A = R - \{3\}$, $B = R - \{1\}$. Let $f : A \to B$ be defined by $f(x)=\frac{x-2}{x-3}$. Then, show that f is bijective.

34. If $\int\left[\log(\log x)+\frac{1}{(\log x)^2}\right]dx = x[f(x)-g(x)]+C$, then find $f(x)$ and $g(x)$.

OR

Evaluate : $\int \frac{1}{\sin^4 x+\sin^2 x\cos^2 x+\cos^4 x}dx$

35. If $f(x) = x^n$, then find the value of

$$f(1)-\frac{f'(1)}{1!}+\frac{f''(1)}{2!}-\frac{f'''(1)}{3!}+.....+\frac{(-1)^n f^n(1)}{n!}.$$

SECTION-V

All questions are compulsory. In case of internal choices attempt any one.

36. Find the shortest distance between the lines $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ and $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$.

OR

Show that the lines $\vec{r}=(\hat{i}+\hat{j}-\hat{k})+\lambda(3\hat{i}-\hat{j})$ and $\vec{r}=(4\hat{i}-\hat{k})+\mu(2\hat{i}+3\hat{k})$ are coplanar.

Also, find the equation of the plane containing them.

37. Find a 2×2 matrix B such that $B\begin{bmatrix}1 & -2\\1 & 4\end{bmatrix}=\begin{bmatrix}6 & 0\\0 & 6\end{bmatrix}$.

OR

If $A=\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}$ and I is the identity matrix of order 2, then show that $A^2 = 4A - 3I$. Hence find A^{-1}.

38. Maximize $Z = 4x + 6y$, subject to $3x + 2y \leq 12$, $x + y \geq 4$, $x, y \geq 0$.

OR

$Z = 6x_1 + 2x_2$, subject to $5x_1 + 9x_2 \leq 90$, $x_1 + x_2 \geq 4$, $x_2 \leq 8$, $x_1 \geq 0$, $x_2 \geq 0$. Find the minimum value of Z.

SOLUTIONS

1. Here, $R = \{(2, 1), (3, 1)\}$ $\therefore$ Range of $R = \{1\}$

2. We have, $A=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$

$\therefore$ adj $(A)=\begin{bmatrix}4 & -3\\-2 & 1\end{bmatrix}^T=\begin{bmatrix}4 & -2\\-3 & 1\end{bmatrix}$

OR

We have, $\begin{vmatrix}\cos 15° & \sin 15°\\ \sin 75° & \cos 75°\end{vmatrix}$

$= \cos 75° \cos 15° - \sin 75° \sin 15°$

$= \cos(75° + 15°) = \cos 90° = 0$

3. We know that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow \cos^2\alpha + \cos^2(90° - \alpha) + \cos^2\gamma = 1$

$\Rightarrow \cos^2\alpha + \sin^2\alpha + \cos^2\gamma = 1 \Rightarrow 1 + \cos^2\gamma = 1$

$\Rightarrow \cos^2\gamma = 0 \Rightarrow \cos\gamma = 0 \Rightarrow \gamma = 90°$

OR

If l, m, n are the direction cosines of the line, then equation of the line which passes through (x_1, y_1, z_1) is $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$.

4. Here, $a_{11}=\frac{(1+2\times1)^2}{2}=\frac{9}{2}$, $a_{12}=\frac{(1+2\times2)^2}{2}=\frac{25}{2}$,

$a_{21}=\frac{(2+2\times1)^2}{2}=8$ and $a_{22}=\frac{(2+2\times2)^2}{2}=18$

So, the required matrix $A=\begin{bmatrix}9/2 & 25/2\\8 & 18\end{bmatrix}$

5. $\int \frac{2-3\sin x}{\cos^2 x}dx = \int\left(\frac{2}{\cos^2 x} - \frac{3\sin x}{\cos^2 x}\right)dx$

$= \int(2\sec^2 x - 3\sec x \tan x)dx$ = 2tan$x$ – 3sec$x$ + C

6. For transitivity of a relation, if $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$

We have, $R = \{(1, 2), (2, 1)\}$

$(1, 2) \in R$ and $(2, 1) \in R$ but $(1, 1) \notin R$

$\therefore$ R is not transitive.

7. Let $\vec{a} = 2\hat{i} - \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$

We know, $\vec{a}$ and $\vec{b}$ are orthogonal iff $\vec{a}\cdot\vec{b} = 0$

$\Rightarrow$ $(2\hat{i} - \lambda\hat{j} + \hat{k})\cdot(\hat{i} + 2\hat{j} - \hat{k}) = 0$

$\Rightarrow$ $2 - 2\lambda - 1 = 0 \Rightarrow 1 - 2\lambda = 0 \Rightarrow \lambda = \frac{1}{2}$

OR

Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$= \frac{(7\hat{i}+\hat{j}-4\hat{k})\cdot(2\hat{i}+6\hat{j}+3\hat{k})}{\sqrt{(2)^2+(6)^2+(3)^2}} = \frac{14+6-12}{7} = \frac{8}{7}$

8. We have, $\int_2^3 3^x\,dx = \left[\frac{3^x}{\log 3}\right]_2^3 = \frac{3^3-3^2}{\log 3} = \frac{18}{\log 3}$

9. The given line is $5x - 3 = 15y + 7 = 3 - 10z$

$\Rightarrow \frac{x-\frac{3}{5}}{\frac{1}{5}} = \frac{y+\frac{7}{15}}{\frac{1}{15}} = \frac{z-\frac{3}{10}}{-\frac{1}{10}}$

Its direction ratios are $\frac{1}{5}, \frac{1}{15}, -\frac{1}{10}$ that are proportional to 6, 2, –3.

Now, $\sqrt{6^2+2^2+(-3)^2} = 7$

$\therefore$ Its direction cosines are $\frac{6}{7}, \frac{2}{7}, -\frac{3}{7}$.

10. The given differential equation is

$(1+x^2) + (2xy - \cot x)\frac{dx}{dy} = 0$

$\Rightarrow (1+x^2)\frac{dy}{dx} + 2xy - \cot x = 0$

$\Rightarrow \frac{dy}{dx} + \frac{2x}{1+x^2}\cdot y = \frac{\cot x}{1+x^2}$

$\therefore$ I.F. $= e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1 + x^2$.

OR

The given differential equation is

$\frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^3\right\} = 0 \Rightarrow 3.\left(\frac{dy}{dx}\right)^2 . \frac{d^2y}{dx^2} = 0$

Order = 2 and Degree = 1

$\therefore$ Order + Degree = 2 + 1 = 3

11. Given, $A = \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}$ $\because$ A is symmetric.

$\therefore$ $A' = A$

$\Rightarrow \begin{bmatrix} 0 & 3 & 3a \\ 2b & 1 & 3 \\ -2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}$

On comparing the corresponding elements of the matrices, we get $a = \frac{-2}{3}$ and $b = \frac{3}{2}$.

OR

Given, $A^T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$\Rightarrow B^T = \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$

$\therefore$ $A^T - B^T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -3 & 0 \\ -1 & -2 \end{bmatrix}$

12. Let $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} - 2p\hat{j} + 3\hat{k}$

For $\vec{a}$ and $\vec{b}$ to be parallel, $\vec{b} = \lambda\vec{a}$.

$\Rightarrow \hat{i} - 2p\hat{j} + 3\hat{k} = \lambda(3\hat{i} + 2\hat{j} + 9\hat{k}) = 3\lambda\hat{i} + 2\lambda\hat{j} + 9\lambda\hat{k}$

$\Rightarrow 1 = 3\lambda; -2p = 2\lambda, 3 = 9\lambda \Rightarrow \lambda = \frac{1}{3}$ and $p = -\lambda = -\frac{1}{3}$

13. Direction ratios of OP are (3–0, 12–0, 4–0) *i.e.*, (3, 12, 4)

Also, $\sqrt{3^2+(12)^2+(4)^2} = 13$

$\therefore$ Direction cosines are

$\left(\frac{3}{13}, \frac{12}{13}, \frac{4}{13}\right)$ or $\left(\frac{-3}{13}, \frac{-12}{13}, \frac{-4}{13}\right)$.

14. Since, C is the mid point of $A(2, -1)$ and $B(-4, 3)$.

$\therefore$ Coordinates of C is $\left(\frac{2-4}{2}, \frac{-1+3}{2}\right) = (-1, 1)$

$\therefore$ $\overrightarrow{OC} = -\hat{i} + \hat{j}$

15. As R is an equivalence relation on set A having n elements.
Hence, R has atleast n ordered pairs.

16. Here, direction cosines of the given line are $\cos\theta_1$, $\cos\theta_2$ and $\cos\theta_3$

$\therefore$ $\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 = 1$ ...(i)

Now, $\cos 2\theta_1 + \cos 2\theta_2 + \cos 2\theta_3$
$= 2(\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3) - 3$ [$\because \cos 2\theta = 2\cos^2\theta - 1$]
$= 2(1) - 3$ [using (i)]
$= -1$

17. **(i) (c) :** If P is the rent price per apartment and N is the number of rented apartment, then the profit is given by, $NP - 500\,N = N(P - 500)$
[$\because$ ₹ 500/month is the maintenance charges for each occupied unit]

(ii) (c) : If x be the number of non-rented apartments, then $N = 50 - x$ and $P = 10000 + 250\,x$
Thus, profit $= N(P - 500) = (50 - x)(10000 + 250\,x - 500)$
$= (50 - x)(9500 + 250\,x) = 250(50 - x)(38 + x)$

(iii) (b) : Clearly, if $P = 10500$, then
$10500 = 10000 + 250\,x \Rightarrow x = 2 \Rightarrow N = 48$

(iv) (a) : Also, if $P = 11000$, then
$11000 = 10000 + 250\,x \Rightarrow x = 4$ and so profit
$P(4) = 250(50 - 4)(38 + 4) =$ ₹ 483000

(v) (b) : We have, $P(x) = 250(50 - x)(38 + x)$
Now, $P'(x) = 250[50 - x - (38 + x)] = 250[12 - 2x]$
For maxima/minima, put $P'(x) = 0$
$\Rightarrow$ $12 - 2x = 0 \Rightarrow x = 6$
Thus, price per apartment is, $P = 10000 + 1500 = 11500$
Hence, the rent that maximizes the profit is ₹ 11500.

18. **(i) (a) :** Clearly, $P(\mathrm{T}_2$ winning a match against $\mathrm{T}_1)$
$= P(\mathrm{T}_1 \text{ losing}) = \frac{1}{5}$

(ii) (d) : Clearly, $P(\mathrm{T}_2$ drawing a match against $\mathrm{T}_1)$
$= P(\mathrm{T}_1 \text{ drawing}) = \frac{3}{10}$

(iii) (d) : According to given information, we have the following possibilities for the values of X and Y.

X	4	3	2	1	0
Y	0	1	2	3	4

Now, $P(X > Y) = P(X = 4, Y = 0) + P(X = 3, Y = 1)$
$= P(\mathrm{T}_1 \text{ win})\,P(\mathrm{T}_1 \text{ win}) + P(\mathrm{T}_1 \text{ win})\,P(\text{match draw})$
$+ P(\text{match draw})\,P(\mathrm{T}_1 \text{ win})$

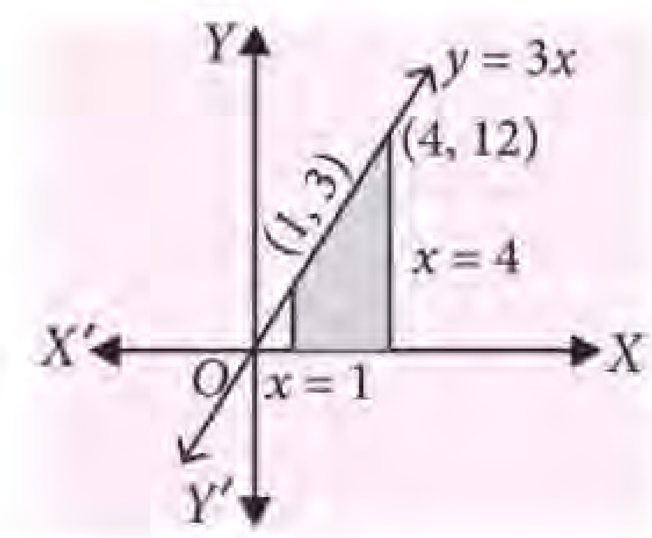
$$= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{3}{10} + \frac{3}{10}\cdot\frac{1}{2} = \frac{5+3+3}{20} = \frac{11}{20}$$

(iv) (c) : $P(X = Y) = P(X = 2, Y = 2)$
$= P(\mathrm{T}_1 \text{ win})\,P(\mathrm{T}_2 \text{ win}) + P(\mathrm{T}_2 \text{ win})\,P(\mathrm{T}_1 \text{ win})$
$+ P(\text{match draw})\,P(\text{match draw})$

$$= \frac{1}{2}\cdot\frac{1}{5} + \frac{1}{5}\cdot\frac{1}{2} + \frac{3}{10}\cdot\frac{3}{10} = \frac{1}{10} + \frac{1}{10} + \frac{9}{100} = \frac{29}{100}$$

(v) (a) : From the given information, it is clear that maximum sum of X and Y can be 4, therefore $P(X + Y = 8) = 0$

19. Area enclosed by line $y = 3x$, x-axis, $x = 1$ and $x = 4$ is shown in figure.



$\therefore$ Required area $= \int_1^4 3x\,dx$

$$= \left[\frac{3x^2}{2}\right]_1^4 = \frac{3}{2}[16 - 1]$$

$$= \frac{3}{2} \times 15 = \frac{45}{2} = 22.5 \text{ sq. units}$$

20. We have, $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - 3\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{4} - 3\left(\frac{\pi}{3}\right)$

$$= \frac{\pi}{4} - \pi = \frac{-3\pi}{4}$$

$\left[\because \text{Principal value of } \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} \text{ and that of } \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}\right]$

OR

We have, $\sin[\cot^{-1}(x + 1)] = \cos(\tan^{-1}x)$ ... (i)
Let $\cot^{-1}(x + 1) = A$ and $\tan^{-1}x = B$

$\Rightarrow$ $x + 1 = \cot A \Rightarrow \sin A = \frac{1}{\sqrt{(x+1)^2 + 1}}$

Also, $x = \tan B \Rightarrow \cos B = \frac{1}{\sqrt{x^2 + 1}}$

Now, $\sin A = \cos B$ [From (i)]

$\Rightarrow$ $\frac{1}{\sqrt{(x+1)^2 + 1}} = \frac{1}{\sqrt{x^2 + 1}} \Rightarrow (x + 1)^2 + 1 = x^2 + 1$

$\Rightarrow$ $1 + 2x = 0 \Rightarrow x = -\frac{1}{2}$

21. Clearly, $P(E) = P(X = 2) + P(X = 3) + P(X = 5) + P(X = 7)$
$= 0.23 + 0.12 + 0.20 + 0.07 = 0.62$
$P(F) = P(X = 1) + P(X = 2) + P(X = 3)$
$= 0.15 + 0.23 + 0.12 = 0.50$

$P(E \cap F) = P(X = 2) + P(X = 3) = 0.23 + 0.12 = 0.35$

$\therefore \quad P(E \cup F) = P(E) + P(F) - P(E \cap F)$
$= 0.62 + 0.50 - 0.35 = 0.77$

22. Given, $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \Rightarrow |A| = 6 + 1 = 7 \neq 0,$

$\therefore \quad A^{-1}$ exists.

Now, $(\text{adj } A) = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

23. Let $I = \int \frac{dx}{3\sin^2 x + 4} = \int \frac{\sec^2 x}{3\tan^2 x + 4\sec^2 x} dx$

$= \int \frac{\sec^2 x}{4 + 7\tan^2 x} dx$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$\therefore \; I = \int \frac{dt}{4 + 7t^2} = \frac{1}{2\sqrt{7}} \tan^{-1}\left(\frac{\sqrt{7}\tan x}{2}\right) + c$

OR

Let $I = \int_{\pi/3}^{\pi/4} (\tan x + \cot x)^2 dx$

$= \int_{\pi/3}^{\pi/4} (\tan^2 x + 2 + \cot^2 x)\, dx = \int_{\pi/3}^{\pi/4} (\sec^2 x + \operatorname{cosec}^2 x) dx$

$= [\tan x - \cot x]_{\pi/3}^{\pi/4} = 1 - 1 - \sqrt{3} + \frac{1}{\sqrt{3}} = \frac{-2}{\sqrt{3}}$

24. We have, R.H.L. (at $x = 2$)

$= \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} ax^2 + b = 4a + b$

L.H.L. (at $x = 2$) $= \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2ax - b) = 4a - b$

and $f(2) = 2$

Since, $f(x)$ is continuous at $x = 2$.

$\therefore \quad 4a + b = 2$ and $4a - b = 2$

Solving, we get $a = \frac{1}{2}, \; b = 0$

Thus, $f(x)$ is continuous at $x = 2$ if $a = \frac{1}{2}$ and $b = 0$.

25. We have, $5\frac{dy}{dx} = e^x y^4 \Rightarrow \frac{5}{y^4} dy = e^x dx$

On integrating both sides, we get

$5\int y^{-4} dy = \int e^x dx \Rightarrow \frac{5y^{-3}}{(-3)} = e^x + c \Rightarrow \frac{-5}{3y^3} = e^x + c$

26. Differentiating $x^{2/3} + y^{2/3} = 2$ with respect to x, we get

$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$

$\therefore$ Slope of the tangent at $(1, 1) = -1$

Also, the slope of the normal at $(1, 1)$ is given by

$\frac{-1}{\text{slope of the tangent at } (1, 1)} = 1$

Therefore, the equation of the normal at $(1, 1)$ is

$y - 1 = 1(x - 1) \Rightarrow y - x = 0$

27. Let E be the event that the coin tossed twice shows first head and then tail and F be the event that the coin drawn is fair.

$P(F/E) = \frac{P(F) \cdot P(E/F)}{P(F) \cdot P(E/F) + P(\bar{F}) P(E/\bar{F})}$

$= \frac{\frac{m}{N} \cdot \frac{1}{2} \cdot \frac{1}{2}}{\frac{m}{N} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{N - m}{N} \cdot \frac{2}{3} \cdot \frac{1}{3}} = \frac{m/4}{m/4 + 2(N - m)/9}$

$= \frac{9m}{m + 8N}$

28. We have, $\vec{a}$ and $\vec{b}$ are collinear.

$\therefore \; \vec{a} = \lambda\vec{b} \Rightarrow -3\hat{i} + n\hat{j} + 4\hat{k} = \lambda(-2\hat{i} + 4\hat{j} + p\hat{k})$

$\Rightarrow \lambda = \frac{3}{2}$

Also, $n = 4\lambda \Rightarrow n = 4 \times \frac{3}{2} = 6$

And, $\lambda p = 4 \Rightarrow p = 4 \times \frac{2}{3} = \frac{8}{3}$

OR

We have, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\hat{i} + \hat{j}$

Now, $(\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$

$\Rightarrow \quad \lambda\vec{a} + \mu\vec{b} = \hat{i}(0 - 0) - \hat{j}(0 - 0) + \hat{k}(0 - 1)$

$\Rightarrow \quad (\lambda + \mu)\hat{i} + (\lambda + \mu)\hat{j} + (\lambda)\hat{k} = -\hat{k}$

On comparing, we get $\lambda = -1$ and $\lambda + \mu = 0$

29. Since, $f(x) = \tan^{-1}(\sin x + \cos x)$

$\therefore \quad f'(x) = \frac{1}{1 + (\sin x + \cos x)^2}(\cos x - \sin x)$

$= \frac{\sqrt{2}\cos\left(x + \frac{\pi}{4}\right)}{1 + (\sin x + \cos x)^2}$

$f(x)$ is increasing, if $f'(x) > 0 \Rightarrow \cos\left(x + \frac{\pi}{4}\right) > 0$

$\Rightarrow \quad -\frac{\pi}{2} < x + \frac{\pi}{4} < \frac{\pi}{2} \Rightarrow -\frac{3\pi}{4} < x < \frac{\pi}{4}$

30. Given, $\frac{dy}{dx} = \frac{2}{x+y}$

Put $x + y = z \Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$

So, given equation becomes

$$\frac{dz}{dx} - 1 = \frac{2}{z} \Rightarrow \frac{dz}{dx} = \frac{z+2}{z}$$

$$\Rightarrow dx = \frac{zdz}{z+2} = \left(1 - \frac{2}{z+2}\right)dz$$

On integrating, we get

$$x + c_1 = z - 2\ln(z+2) \Rightarrow \ln(x+y+2) = \frac{y - c_1}{2}$$

$$\Rightarrow x + y + 2 = ce^{\frac{y}{2}}$$

31. Given, $f(x) = \begin{cases} \sin x, & \text{for } x \geq 0 \\ 1 - \cos x, & \text{for } x \leq 0 \end{cases}$ and $g(x) = e^x$

$$\therefore \quad gof(x) = \begin{cases} e^{\sin x}, & x \geq 0 \\ e^{1-\cos x}, & x \leq 0 \end{cases}$$

$\therefore$ L.H.D. $= (gof)'(0 - h) = \lim_{h\to 0} \frac{gof(0-h) - gof(h)}{-h}$

$$= \lim_{h\to 0} \frac{e^{1-\cos(0-h)} - e^{1-\cos h}}{-h} = 0$$

R.H.D. $= (gof)'(0 + h)$

$$= \lim_{h\to 0} \frac{gof(0+h) - gof(h)}{h} = \lim_{h\to 0} \frac{e^{\sin h} - e^{\sin h}}{h} = 0$$

$\therefore$ R.H.D. = L.H.D. $= 0 \Rightarrow (gof)'(0) = 0$

OR

Let $y_1 = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = 2\tan^{-1} x$

and $y_2 = \cot^{-1}\left(\frac{1-3x^2}{3x-x^3}\right) = 3\tan^{-1} x$

Differentiating w.r.t. x, we get

$$\frac{dy_1}{dx} = \frac{2}{1+x^2} \text{ and } \frac{dy_2}{dx} = \frac{3}{1+x^2}$$

$$\Rightarrow \frac{dy_1}{dy_2} = \frac{\left(\frac{dy_1}{dx}\right)}{\left(\frac{dy_2}{dx}\right)} = \frac{\left(\frac{2}{1+x^2}\right)}{\left(\frac{3}{1+x^2}\right)} = \frac{2}{3}$$

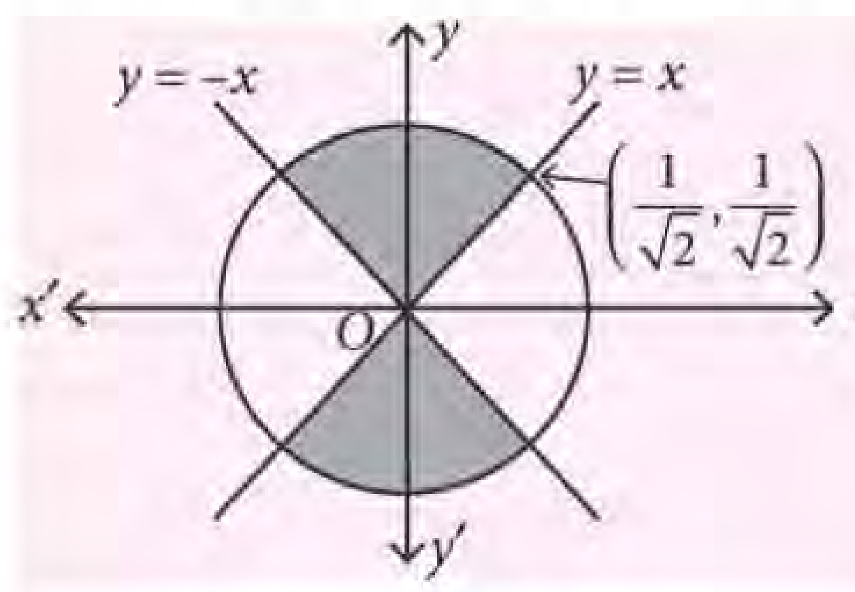
32. We have, $y = x$ ...(i)

$y = -x$ ...(ii)

$x^2 + y^2 = 1$ ...(iii)

Solving (i) and (iii), we get $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\therefore$ Required area = Area of the shaded region



= 4 (Area of the shaded region in first quadrant)

$$= 4\int_0^{1/\sqrt{2}} (\sqrt{1-x^2} - x)\,dx$$

$$= 4\left[\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x - \frac{x^2}{2}\right]_0^{1/\sqrt{2}}$$

$$= 4\left[\frac{1}{2\sqrt{2}} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{\pi}{4} - \frac{1}{4}\right] = \frac{\pi}{2} \text{ sq. units}$$

33. Let x and y be two arbitrary elements in A.

Then, $f(x) = f(y) \Rightarrow \frac{x-2}{x-3} = \frac{y-2}{y-3}$

$\Rightarrow \quad xy - 3x - 2y + 6 = xy - 3y - 2x + 6$

$\Rightarrow \quad x = y, \forall\, x, y \in A$

So, f is an injective mapping.

Again, let y be an arbitrary element in B, then $f(x) = y$

$$\Rightarrow \frac{x-2}{x-3} = y \Rightarrow x = \frac{3y-2}{y-1}$$

Clearly, $\forall\, y \in B$, there exist $x = \frac{3y-2}{y-1} \in A$ such that

$$f(x) = f\left(\frac{3y-2}{y-1}\right) = \frac{\frac{3y-2}{y-1} - 2}{\frac{3y-2}{y-1} - 3} = y$$

Thus, every element in the co-domain B has its pre-image in A, so f is a surjective. Hence, $f : A \to B$ is bijective.

Monthly Test Drive CLASS XII ANSWER KEY

1.	(d)	2.	(b)	3.	(b)	4.	(a)	5.	(b)
6.	(a)	7.	(a, c)	8.	(a)	9.	(b, c)	10.	(a,c)
11.	(b, c)	12.	(b,c)	13.	(a,c)	14.	(c)	15.	(b)
16.	(c)	17.	(0)	18.	(2)	19.	(4)	20.	(2)

34. Consider, $\int\left[\log(\log x)+\frac{1}{(\log x)^2}\right]dx$

$$= \int 1\cdot\log(\log x)dx + \int\frac{1}{(\log x)^2}\,dx$$

$$=\log(\log x)\cdot x-\int\left(\frac{1}{(\log x)\times x}\times x\right)dx+\int\frac{1}{(\log x)^2}\,dx+C$$

$$=x\log(\log x)-\int\frac{1}{\log x}\,dx+\int\frac{1}{(\log x)^2}\,dx+C$$

$$=x\log(\log x)-\frac{1}{\log x}\times x+\int\frac{-1}{(\log x)^2}\cdot\frac{x}{x}dx+\int\frac{1}{(\log x)^2}\,dx+C$$

$$=x\log(\log x)-\frac{x}{\log x}-\int\frac{1}{(\log x)^2}dx+\int\frac{1}{(\log x)^2}dx+C$$

$$=x\left[\log(\log x)-\frac{1}{\log x}\right]+C=x\left[f(x)-g(x)\right]+C$$

$\therefore\quad f(x)=\log(\log x),\ g(x)=\frac{1}{\log x}$

OR

Refer to Answer 46, page no. 159-160 of MTG CBSE Champion Mathematics, Class 12.

35. Since, $f(x) = x^n \Rightarrow f(1) = 1$

$f'(x) = nx^{n-1} \Rightarrow f'(1) = n$

$f''(x) = n(n-1)x^{n-2} \Rightarrow f''(1) = n(n-1)$

... ...

... ...

$f^n(x) = [n(n-1)(n-2)\ ...\ 2\cdot 1]\ x^{n-n}$

$\Rightarrow\ f^n(1) = n(n-1)(n-2)\ ...\ 2\cdot 1$

We have,

$$f(1)-\frac{f'(1)}{1!}+\frac{f''(1)}{2!}-\frac{f'''(1)}{3!}+.....+\frac{(-1)^n f^n(1)}{n!}$$

$$=1-\frac{n}{1!}+\frac{n(n-1)}{2!}-\frac{n(n-1)(n-2)}{3!}+.....+\frac{(-1)^n n(n-1)(n-2)....2\cdot 1}{n!}$$

$= 1 - {}^nC_1 + {}^nC_2 - {}^nC_3 + ... + (-1)^n\, {}^nC_n = (1-1)^n = 0$

36. We have, $(x_1, y_1, z_1) = (3, 8, 3)$, $(x_2, y_2, z_2) = (-3, -7, 6)$, $(a_1, b_1, c_1) = (3, -1, 1)$ and $(a_2, b_2, c_2) = (-3, 2, 4)$.

Now, shortest distance

$$=\frac{\begin{vmatrix}(x_2-x_1) & (y_2-y_1) & (z_2-z_1)\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-a_1c_2)^2+(a_1b_2-b_1a_2)^2}}$$

Here, numerator $=\begin{vmatrix}-6 & -15 & 3\\ 3 & -1 & 1\\ -3 & 2 & 4\end{vmatrix}$

$= -6(-4-2) + 15\,(12+3) + 3\,(6-3)$

$= 36 + 225 + 9 = 270$

$\therefore$ Required distance

$$=\frac{270}{\sqrt{(-4-2)^2+(-3-12)^2+(6-3)^2}}$$

$$=\frac{270}{\sqrt{270}}=\sqrt{270}=3\sqrt{30}$$

OR

Refer to Answer 76, page no. 272 of MTG CBSE Champion Mathematics, Class 12.

37. Let $A=\begin{bmatrix}1 & -2\\ 1 & 4\end{bmatrix}$ and $C=\begin{bmatrix}6 & 0\\ 0 & 6\end{bmatrix}$

Then, $|A|=\begin{vmatrix}1 & -2\\ 1 & 4\end{vmatrix}=6\neq 0$

So, A is invertible.

The given matrix equation is

$$B\begin{bmatrix}1 & -2\\ 1 & 4\end{bmatrix}=\begin{bmatrix}6 & 0\\ 0 & 6\end{bmatrix}$$

$\Rightarrow\ BA = C\ \Rightarrow\ (BA)A^{-1} = CA^{-1}$

$\Rightarrow\ B(AA^{-1}) = CA^{-1} \Rightarrow BI = CA^{-1} \Rightarrow B = CA^{-1}$

$\therefore\ \operatorname{adj} A=\begin{bmatrix}4 & -1\\ 2 & 1\end{bmatrix}^T=\begin{bmatrix}4 & 2\\ -1 & 1\end{bmatrix}$

$\Rightarrow\ A^{-1}=\frac{1}{|A|}\operatorname{adj}A=\frac{1}{6}\begin{bmatrix}4 & 2\\ -1 & 1\end{bmatrix}$

Now, $B = CA^{-1}$

$\Rightarrow\ B=\begin{bmatrix}6 & 0\\ 0 & 6\end{bmatrix}\cdot\frac{1}{6}\begin{bmatrix}4 & 2\\ -1 & 1\end{bmatrix}=\frac{1}{6}\begin{bmatrix}6 & 0\\ 0 & 6\end{bmatrix}\begin{bmatrix}4 & 2\\ -1 & 1\end{bmatrix}$

$\Rightarrow\ B=\frac{1}{6}\begin{bmatrix}24+0 & 12+0\\ 0-6 & 0+6\end{bmatrix}=\begin{bmatrix}4 & 2\\ -1 & 1\end{bmatrix}$

OR

Refer to Answer 95, page no. 70 of MTG CBSE Champion Mathematics, Class 12.

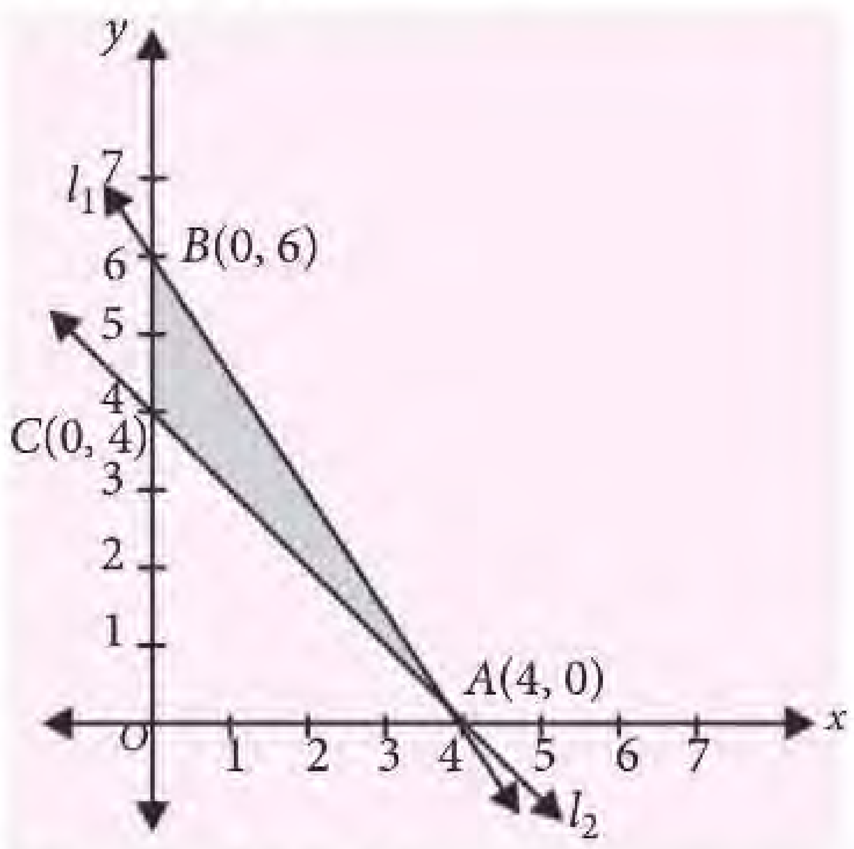
38. We have, maximize $Z = 4x + 6y$

Subject to $3x + 2y \le 12$, $x + y \ge 4$, $x, y \ge 0$

Let $l_1 : 3x + 2y = 12$; $l_2 : x + y = 4$

$l_3 : x = 0$ and $l_4 : y = 0$

Shaded portion ABC is the feasible region, where $A(4, 0)$, $B(0, 6)$, $C(0, 4)$.



Now maximize $Z = 4x + 6y$

Z at $A(4, 0) = 4(4) + 6(0) = 16$

Z at $B(0, 6) = 4(0) + 6(6) = 36$

Z at $C(0, 4) = 4(0) + 6(4) = 24$

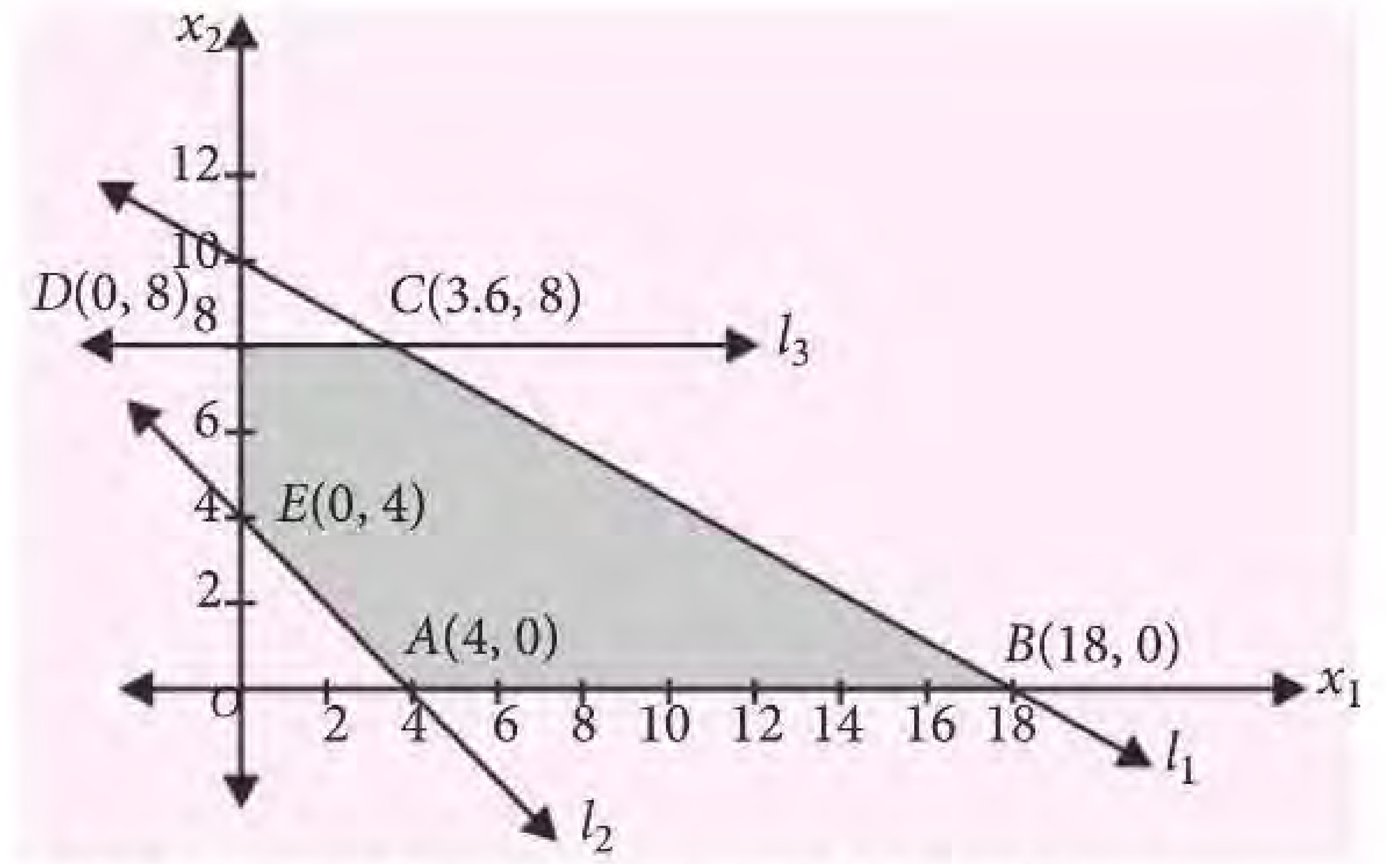
Thus, Z is maximized at $B(0, 6)$ and its maximum value is 36.

OR

We have, minimize $Z = 6x_1 + 2x_2$

Subject to $5x_1 + 9x_2 \leq 90$, $x_1 + x_2 \geq 4$, $x_2 \leq 8$, $x_1, x_2 \geq 0$

Let $l_1 : 5x_1 + 9x_2 = 90$, $l_2 : x_1 + x_2 = 4$, $l_3 : x_2 = 8$, $l_4 : x_1 = 0$ and $l_5 : x_2 = 0$



For C : Solving l_1 and l_3, we get $C(3.6, 8)$

Shaded portion $ABCDE$ is the feasible region, where $A(4, 0)$, $B(18, 0)$, $C(3.6, 8)$, $D(0, 8)$, $E(0, 4)$.

Now, minimize $Z = 6x_1 + 2x_2$

Z at $A(4, 0) = 6(4) + 2(0) = 24$

Z at $B(18, 0) = 6(18) + 2(0) = 108$

Z at $C(3.6, 8) = 6(3.6) + 2(8) = 37.6$

Z at $D(0, 8) = 6(0) + 2(8) = 16$

Z at $E(0, 4) = 6(0) + 2(4) = 8$

Thus, Z is minimized at $E(0, 4)$ and its minimum value is 8.

MONTHLY TEST DRIVE

XII

This specially designed column enables students to self analyse their extent of understanding of all chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. A plane which is perpendicular to two planes $2x - 2y + z = 0$ and $x - y + 2z = 4$, passes through $(1, -2, 1)$. The distance (in units) of the plane from the point $(1, 2, 2)$ is
   (a) 0 (b) 1 (c) $\sqrt{2}$ (d) $2\sqrt{2}$

2. The area of the region between the curves $y=\sqrt{\dfrac{1+\sin x}{\cos x}}$ and $y=\sqrt{\dfrac{1-\sin x}{\cos x}}$ bounded by the lines $x = 0$ and $x=\dfrac{\pi}{4}$ is
   (a) $\displaystyle\int_0^{\sqrt{2}-1} \frac{t}{(1+t^2)\sqrt{1-t^2}}dt$
   (b) $\displaystyle\int_0^{\sqrt{2}-1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}dt$
   (c) $\displaystyle\int_0^{\sqrt{2}+1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}dt$
   (d) $\displaystyle\int_0^{\sqrt{2}+1} \frac{t}{(1+t^2)\sqrt{1-t^2}}dt$
   where $t=\tan\dfrac{x}{2}$.

3. The value of $\tan\left(\cos^{-1}\dfrac{4}{5}+\tan^{-1}\dfrac{2}{3}\right)$ is
   (a) $\dfrac{6}{17}$ (b) $\dfrac{17}{6}$
   (c) $\dfrac{16}{7}$ (d) $\dfrac{7}{16}$

4. A problem in mathematics is given to three students A, B, C and their respective probability of solving the problem is $\dfrac{1}{2}, \dfrac{1}{3}$ and $\dfrac{1}{4}$. Probability that the problem is solved is
   (a) $\dfrac{3}{4}$ (b) $\dfrac{1}{2}$ (c) $\dfrac{2}{3}$ (d) $\dfrac{1}{3}$

5. The value of the integral $I=\displaystyle\int_0^1 x(1-x)^n\,dx$ is
   (a) $\dfrac{1}{n+2}$ (b) $\dfrac{1}{n+1}-\dfrac{1}{n+2}$
   (c) $\dfrac{1}{n+1}+\dfrac{1}{n+2}$ (d) $\dfrac{1}{n+1}$

6. If f is a differentiable function satisfying $f\left(\dfrac{1}{n}\right)=0$ for all $n \ge 1, n \in I$, then
   (a) $f'(0) = 0 = f(0)$
   (b) $|f(x)| \le 1, x \in (0, 1)$
   (c) $f(x) = 0, x \in (0, 1]$
   (d) $f(0) = 0$ but $f'(0)$ not necessarily zero

One or More Than One Option(s) Correct Type

7. For any two events A and B in a sample space,
   (a) $P(A/B) \ge \dfrac{P(A)+P(B)-1}{P(B)}, P(B) \ne 0$, is always true
   (b) $P(A \cap \overline{B}) = P(A) - P(A \cap B)$, does not hold
   (c) $P(A \cup B) = 1 - P(\overline{A})\,P(\overline{B})$, if A and B are independent
   (d) $P(A \cup B) = 1 - P(\overline{A})\,P(\overline{B})$, if A and B are disjoint

8. The value(s) of $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$ is (are)

(a) $\frac{22}{7}-\pi$ (b) $\frac{2}{105}$ (c) 0 (d) $\frac{71}{15}-\frac{3\pi}{2}$

9. Which of the following functions are continuous on $(0, \pi)$?

(a) $\tan x$ (b) $\int_0^x t \sin\frac{1}{t}dt$

(c) $\begin{cases} 1, & 0 < x \le \frac{3\pi}{4} \\ 2\sin\frac{2}{9}x, & \frac{3\pi}{4} < x < \pi \end{cases}$

(d) $\begin{cases} x\sin x, & 0 < x \le \frac{\pi}{2} \\ \frac{\pi}{2}\sin(\pi + x), & \frac{\pi}{2} < x < \pi \end{cases}$

10. Let $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}, \vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{c} = \hat{i} + \hat{j} - 2\hat{k}$ be three vectors. A vector in the plane of $\vec{b}$ and $\vec{c}$ whose projection on $\vec{a}$ is of magnitude $\sqrt{\frac{2}{3}}$ is

(a) $2\hat{i} + 3\hat{j} - 3\hat{k}$ (b) $2\hat{i} + 3\hat{j} + 3\hat{k}$

(c) $-2\hat{i} - \hat{j} + 5\hat{k}$ (d) $2\hat{i} + \hat{j} + 5\hat{k}$

11. $f(x)$ is cubic polynomial with $f(2) = 18$ and $f(1) = -1$. Also $f(x)$ has local maxima at $x = -1$ and $f(x)$ has local minima at $x = 0$, then

(a) the distance between $(-1, 2)$ and $(a, f(a))$, where $x = a$ is the point of local minima is $2\sqrt{5}$.

(b) $f(x)$ is increasing for $x \in [1, 2\sqrt{5}]$.

(c) $f'(x)$ has local minima at $x = 1$.

(d) $f(0) = 15$

12. Let $g(x)$ be a function defined on $[-1, 1]$. If the area of the equilateral triangle with two of its vertices at $(0, 0)$ and $[x, g(x)]$ is $\frac{\sqrt{3}}{4}$, then the function $g(x)$ is

(a) $g(x) = -\sqrt{1+x^2}$ (b) $g(x) = \sqrt{1-x^2}$

(c) $g(x) = -\sqrt{1-x^2}$ (d) $g(x) = \sqrt{1+x^2}$

13. The differential equation representing the family of curves $y^2 = 2c(x + \sqrt{c})$, where c is a positive parameter, is of

(a) order 1 (b) order 2

(c) degree 3 (d) degree 4

Comprehension Type

Consider the polynomial $f(x) = 1 + 2x + 3x^2 + 4x^3$. Let s be the sum of all distinct real roots of $f(x)$ and let $t = |s|$.

14. The real numbers lies in the interval

(a) $\left(-\frac{1}{4}, 0\right)$ (b) $\left(-11, -\frac{3}{4}\right)$

(c) $\left(-\frac{3}{4}, -\frac{1}{2}\right)$ (d) $\left(0, \frac{1}{4}\right)$

15. The function $f'(x)$ is

(a) increasing in $\left(-t, -\frac{1}{4}\right)$ and decreasing in $\left(-\frac{1}{4}, 1\right)$

(b) decreasing in $\left(-t, -\frac{1}{4}\right)$ and increasing in $\left(-\frac{1}{4}, t\right)$

(c) increasing in $(-t, t)$ (d) decreasing in $(-t, t)$

Matrix Match Type

16. Consider the following linear equations
$ax + by + cz = 0$, $bx + cy + az = 0$ and $cx + ay + bz = 0$
Match the conditions/expressions in Column I with statements in Column II.

	Column I		Column II
P.	$a + b + c \neq 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$	1.	the equations represent planes meeting only at a single point
Q.	$a + b + c = 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$	2.	the equations represent the line $x = y = z$
R.	$a + b + c \neq 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$	3.	the equations represent identical planes.
S.	$a + b + c = 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$	4.	the equations represent the whole of the three dimensional space.

	P	Q	R	S
(a)	2	1	3	4
(b)	3	4	2	1
(c)	3	2	1	4
(d)	1	2	3	4

Numerical Value Type

17. Let $y'(x) + y(x)g'(x) = g(x)g'(x)$, $y(0) = 0$, $x \in R$, where $f'(x)$ denotes $\dfrac{d\,f(x)}{dx}$ and $g(x)$ is a given non-constant differentiable function on R with $g(0) = g(2) = 0$. Then the value of $y(2)$ is ______.

18. Let f be a function defined on R (the set of all real numbers) such that $f'(x) = 2010(x - 2009)(x - 2010)^2(x - 2011)^3(x - 2012)^4$, for all $x \in R$. If g is a function defined on R with values in the interval $(0, \infty)$ such that $f(x) = \ln(g(x))$, for all $x \in R$, then the number of points in R at which g has a local maximum is _______.

19. Let k be a positive real number and let

$$A = \begin{bmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} 0 & 2\sqrt{k-1} & \sqrt{k} \\ 1-2\sqrt{k} & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{bmatrix}.$$

If det (adj A) + det(adj B) = 10^6, then $[k]$ is equal to _______.

[**Note :** adj M denotes the adjoint of a square matrix M and $[k]$ denotes the largest integer less than or equal to k].

20. The value of $\displaystyle\int_0^1 4x^3 \left\{\frac{d^2}{dx^2}(1-x^2)^5\right\} dx$ is _______.

◈◈

Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted ......
No. of questions correct ......
Marks scored in percentage ......

Check your score! If your score is

> 90%	**EXCELLENT WORK !**	You are well prepared to take the challenge of final exam.
90-75%	**GOOD WORK !**	You can score good in the final exam.
74-60%	**SATISFACTORY !**	You need to score more next time.
< 60%	**NOT SATISFACTORY!**	Revise thoroughly and strengthen your concepts.

YOU ASK WE ANSWER

Do you have a question that you just can't get answered? Use the vast expertise of our MTG team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough. The best questions and their solutions will be printed in this column each month.

1. If $\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}/2$ and $\vec{A},\vec{B},\vec{C}$ are unit vectors then find the angle between $\vec{A}$ and $\vec{C}$.

(Aditya, Delhi)

Ans. Given, $\vec{A}\times(\vec{B}\times\vec{C})=\dfrac{\vec{B}}{2}$

Also, $\vec{A}\times(\vec{B}\times\vec{C})=(\vec{A}\cdot\vec{C})\vec{B}-(\vec{A}\cdot\vec{B})\vec{C}=\dfrac{\vec{B}}{2}$

On comparing coefficients, we have

$$(\vec{A}\cdot\vec{C})\vec{B}=\frac{\vec{B}}{2} \quad\text{and}\quad (\vec{A}\cdot\vec{B})\vec{C}=0$$

$$\Rightarrow\quad \vec{A}\cdot\vec{C}=\frac{1}{2}\ \Rightarrow\ |\vec{A}||\vec{C}||\cos\theta|=\frac{1}{2}\ \Rightarrow\ \theta=\frac{\pi}{3}$$

2. Let $a_1, a_2, \ldots$ be positive numbers in G.P.. For each n, let A_n, G_n, H_n be the arithmetic mean, geometric mean and harmonic mean of $a_1, a_2, \ldots, a_n$. Express the geometric mean of $G_1, G_2, \ldots, G_n$ in terms of $A_1, A_2, \ldots, A_n, H_1, H_2, \ldots, H_n$.

(Khushi, Hyderabad)

Ans. Let the G.P. be $a_1, a_1r, a_1r^2, \ldots, a_1r^{n-1}$

$$A_n=\frac{a_1+a_2+\ldots+a_n}{n}=\frac{a_1}{n}\ (1+r+r^2+\ldots+r^{n-1})$$

$$=\frac{a_1}{n}\left(\frac{1-r^n}{1-r}\right)$$

$$G_n=(a_1a_2\ldots a_n)^{\frac{1}{n}}=\left(a_1^n r^{1+2+\ldots+(n-1)}\right)^{\frac{1}{n}}=a_1r^{(n-1)/2}$$

$$H_n=\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}}=\frac{n}{\frac{1}{a_1}\left(1+\frac{1}{r}+\ldots+\frac{1}{r^{n-1}}\right)}$$

$$=na_1\cdot r^{n-1}\left(\frac{1-r}{1-r^n}\right)$$

$$\therefore\quad A_nH_n=a_1^2r^{(n-1)}=G_n^2 \qquad \ldots(1)$$

The G.M. of $G_1, G_2, \ldots, G_n$ is $(G_1G_2\ldots G_n)^{\frac{1}{n}}$

$$=(A_1H_1\cdot A_2H_2\ldots A_nH_n)^{\frac{1}{2n}}$$

$$=(A_1A_2\ldots A_n\times H_1H_2\ldots H_n)^{1/2n}.$$

3. Let $\Delta_r=\begin{vmatrix} r-1 & n & 6\\ (r-1)^2 & 2n^2 & 4n-2\\ (r-1)^3 & 3n^3 & 3n^2-3n\end{vmatrix}$.

Find the value of $\sum\limits_{r=1}^{n}\Delta_r$. *(Anjana, Patna)*

Ans. $$\sum_{r=1}^{n}\Delta_r=\begin{vmatrix} \sum\limits_{r=1}^{n}(r-1) & n & 6\\ \sum\limits_{r=1}^{n}(r-1)^2 & 2n^2 & 4n-2\\ \sum\limits_{r=1}^{n}(r-1)^3 & 3n^3 & 3n^2-3n\end{vmatrix}$$

$$=\begin{vmatrix} \frac{n(n-1)}{2} & n & 6\\ \frac{(n-1)n(2n-1)}{6} & 2n^2 & 2(2n-1)\\ \frac{(n-1)^2n^2}{4} & 3n^3 & 3n(n-1)\end{vmatrix}$$

$$=\frac{1}{48}\begin{vmatrix} (n-1)n & 2n & 12\\ (n-1)n(2n-1) & 12n^2 & 12(2n-1)\\ (n-1)^2n^2 & 12n^3 & 12n(n-1)\end{vmatrix}$$

By taking factors $(n-1)\,n$ from C_1, $2n$ from C_2 and 12 from C_3, we get

$$\sum_{r=1}^{n}\Delta_r=\frac{1}{2}(n-1)n^2\begin{vmatrix} 1 & 1 & 1\\ 2n-1 & 6n & 2n-1\\ (n-1)n & 6n^2 & n(n-1)\end{vmatrix}=0$$

◈◈